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In Quantum Field Theory Fermion Number Fractionization

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ABSTRACT

The invaluable role played by Dirac equation in some areas of modern quantum field theory is pointed out. In particular its role in the phenomenon of fermion number fractionization is discussed in some detail. Using the inbuilt super symmetry of the Dirac operator. Dirac equation is shown to be exactly solvable in case of various different types of potentials. Some open problems are also pointed out.

Key words:-(fermion, super-symmetry, fractionization, chiral anomalies, kink, antikink)

1.Introduction

In general and elementary particle physics in particular. This equation which is applicable to particles with spin 1/2 (ħ=c=1) is given by

$$\left(i\gamma_{\mu}\partial^{\mu} - m\right)\Psi = 0 \qquad (1.1)$$

This equation immediately predicated the existence of antiparticle of electron i.e. positron with the same mass as electron but equal and opposite electric charge. Soon it was realized that this prediction s valid not only for electron but for any spin – 1/2 particle and today we of course know that it is true for any elementary particle. Another remarkable prediction from Dirac equation follows when one considers the motion of an electron in external electromagnetic field given by

$$\left[i\gamma_{\mu}(\partial^{\mu} - i\, e\, A^{\mu}) - m\right]\Psi = 0 \qquad (1.2)$$

In this way one was lead to the celebrated problem[2] of the Dirac particle in a Coulomb field. As we all know the predicted spectrum is in remarkable agreement with experiment but for the Lamb shift and hyperfine spliting[3]. In some sense it is this disagreement which finally lead to the remarkable discovery of quantum electrodynamics-one of the most successful theory ever discovered by mankind.

The Dirac operator of eq. (2) i.e.,

$$D = \gamma_\mu (\partial^\mu - i\, e\, A^\mu) \qquad (1.3)$$

Which plays a crucial role in many situations, has an inbuilt super-symmetry. In particular, on defining the SUSY charges[4]

$$Q_\pm = \frac{1}{2}(1 \pm \gamma_5) D \qquad (1.4)$$

it easily follows that the operators *Q. Q, and H(=D²)* satisfy the *N=1* super symmetry algebra

$$\{Q_+, Q\} = H; [H, Q_+] = 0, Q_+^2 = Q^2 = 0 \qquad (1.5)$$

This chiral super- symmetry was first successfully exploited in the context of the study of chiral anomalies. Recently we[4] have discovered that the Dirac operator has another super -symmetry (the so called complex super-symmetry) in four and higher even dimensions.

In recent times there has been a renewed interest[5] in understanding the zero (energy) modes and also complete spectrum of the Dirac operator (1.3) for an Euclidean fermionic theory interacting with the background gauge fields. The point is that the zero modes in turn are intimately related to chiral anomalies[6], Witten index, fermion number fractonization[7] etc.

In this paper I have decided to focus on the solution of the Dirac equation (2) in the case of various different situations. Further, as a concrete application of these solutions (in particular the zero modes) I discuss the phenomenon of fermion number fractionization which occurs whenever there are zero modes of the Dirac equation in the background of the topologically nontrivial objects. I also point out the possible experimental relevance of this phenomenon, and finally point out number of open problems.

The plan is the following: In Sec. II first discuss the topologically nontrivial objects. As an illustration, I discuss in details the kink solution in 1+1 dimension. In Sec. III I discuss the solution of the Dirac equation in the background of the kink solution. In particular I show that these equations can be decoupled and that whenever a Schrödinger problem is exactly solvable then always exist a corresponding solvable Dirac problem. In Sec. IV I discuss the quantization of Dirac field and show that in view of the zero energy modes the solution acquires fractional charge. In Sec. V I discuss the experimental relevance of this result. In Sec. VI I discuss the solution of the Dirac equation in other situations and point out some open problems.

2.Topologically Nontrivial Objects: Kink in 1+1

By now we know many topologically nontrivial examples in field theory. Some of them are; kink in 1+1, charged and neutral vortices in 2+1, t'Hooft -Polyakov monopole[10] and dyon[11] in 3+1, Skyrmion[12] in 3+1, 0(3) σ-model solution[13] etc. All these objects are stable due to nontrivial boundary conditions. More technically all these objects are stable since for them there is a nontrivial mapping from the spacetime manifold on to the group manifold: $\pi_{\eta}(8) \neq 0$. It appears that the topologically nontrival solution is a general feature of nonlinear field theories with degenerate minima. As an illustration of these ideas I now discuss the simplest known such example: kink solution in 1+1 dimensions.

Let us consider the Lagrangian in 1+1 dimensions

$$\mathcal{L}_{\beta} = \frac{1}{2}\partial_{\mu}\phi\,\partial^{\mu} - \frac{1}{2}S^2(\phi) \qquad (2.1)$$

The corresponding field energy is

$$E = \int_{-\infty}^{\infty} H dx = \int_{-\infty}^{\infty} \frac{1}{2} dx \left[\left(\frac{d\phi}{dx}\right)^2\right] + \left(\frac{d\phi}{dt}\right)^2 + S^2(\phi) \qquad (2.2)$$

Clearly $E_{min} = 0\; at\; \phi = \phi_{\min}$ which is the ground state of the system. Here without any loss of generality we have assumed that $S^2(\phi_{min}) = 0$. The field equation which follows from here is

$$\frac{d^2\phi}{dt^2} - \frac{d^2\phi}{dx^2} = -S(\phi)S'(\phi) \qquad (2.3)$$

Which in the case of static solution $\left(\frac{d^2\phi}{dt} = 0\right)$ can be integrated to give

$$\left(\frac{d\phi}{dx}\right)^2 = S^2(\phi) + c \qquad (2.4)$$

Now at $\phi = \phi_{\min}, S^2(\phi_{\min}) = 0$ and hence $C_1 = 0$. Thus the static solution is

$$\int \frac{d\phi}{S(\phi)} = \pm x + c_2 \qquad (2.5)$$

One of the most celebrated example of $S^2(\phi)$ with double minima is

$$\int \frac{d\phi}{S(\phi)} = \pm x + c_2 \qquad (2.6)$$

In which case $\phi_{\min} = \pm\, \mu/\sqrt{\lambda}$. On using (2.6) in (2.5) we find that the static solutions are

$$\phi_k^{\pm}(x) = \pm \frac{\mu}{\sqrt{\lambda}} \tanh\ \mu x \qquad (2.7)$$

Note that these solutions are topologically nontrivial in the sense that as $x \to \infty$, $\phi_k^{+}(x)$ goes to $\mu/\sqrt{\lambda}$ which as $x \to \infty$it goes to the other vacua $-\mu/\sqrt{\lambda}$. Thus we can difine a conserved current

$$j_{\mu} = \varepsilon_{\mu v}\partial^{v}\phi, \partial^{\mu} j_{\mu} = 0 \qquad (2.8)$$

The corresponding conserved charge is

$$Q = \int_{-\infty}^{\infty} j_0(x) d\,x = \phi(x = +\infty) - \phi(x = -\infty) \qquad (2.9)$$

$$= \frac{2\mu}{\sqrt{\lambda}}\; for\; \phi_k^{+}(x)$$

$$= -2\mu/\sqrt{\lambda}\; for\; \phi_k^{(-)}(x)$$

Here conventionally $\phi_k^{+}(x)$ and $\phi_k^{(-)}(x)$ are called kink and antikink respectively. Note that both the solutions are degenerate in energy i.e.

$$E = 4\mu^2/3\lambda \qquad (2.10)$$

It is worth nothing that E as well as ϕ_k are inversely proportional to λ. Since λ is the coefficient of the ϕ^4 term, it clearly shows that these topological solutions are nonperturbative in nature and that they could not have been obtained in perturbation theory.

3.Solution of Dirac Equation in the Kink Background

We shall now consider the solution of the Dirac equation in the background of the topologically nontrivial kink solution. In particular consider

$$\mathcal{L} = \mathcal{L}_B + \overline{\Psi}\left(i\,\gamma_\mu\,\partial^\mu - \lambda\phi\right)\Psi \qquad (3.1)$$

Where $\mathcal{L}_B$is as given by eq. (2.1). For small λ one can neglect the $\Psi - \phi$ coupling and find solution $\mathcal{L}_B$ and then one wants to quantize Dirac system with ϕ taken as external $c - n_0$ background field. As a first step in that direction, one must first obtain the solution of the Dirac equation in the background of the kink solution. The Dirac equation to be considered is

$$\left(i\,\gamma_\mu \partial^\mu - \lambda\phi\right)\Psi(x,t) = 0 \qquad (3.2)$$

Let

$$\Psi(x,t) = e^{-i\omega t}\Psi(x) \qquad (3.3)$$

We shall work with the following representation of the γ-matrices

$$\gamma^0 = \sigma^1, \gamma^1 = i\,\sigma^3 \qquad (3.4)$$

On choosing $\Psi(x) = \begin{pmatrix}\Psi_1 & (x)\\ \Psi_2 & (x)\end{pmatrix}$ it easily follows that the equation for Ψ_1 and Ψ_2 can be decoupled[4]

$$\left[\frac{d^2}{d\,x^7} + \lambda^2\phi^2 - \lambda\,\phi\,(x)\right]\Psi(x) = \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots?$$

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We immediately recognize that the operators acting on the l.h.s……. Of eq. (3.5) are the supersymmetric partner Hamiltonian $H\pm$ defined by

$$H \equiv A\,A, \;\; H = A\,A \qquad (3.6)$$

Where

$$A = \frac{d}{dx} + \lambda\phi(x); \quad A^{+} = \frac{d}{dx} + \lambda\phi(x) \qquad (3.7)$$

The super-symmetry charges $Q \quad Q^4$ are easily written in terms of A and A^+

$$Q = \begin{pmatrix}0 & A\\ 0 & 0\end{pmatrix}.\, Q^4 \begin{pmatrix}0 & 0\\ A & 0\end{pmatrix} \qquad (3.8)$$

$Q, Q^4 \equiv \begin{pmatrix} H & 0 \\ 0 & H_+ \end{pmatrix}$ can now be shown to satisfy the SUSY algebra (1.5). One can now use the whole machinery of supersymmetirc quantum mechanics using which large number of potentials have been solved in nonrelativisitc quantum mechanics[17] and expressions for eigenvalues, eighnfunctions and scattering matrix have been obtained. All that machinery can now also be used for the solution of Dirac eq. (3.2) when the kink solution $\lambda\,\phi$ (x) is to be indentified with the super potential in the Schrödinger problem. In this way, we conclude that if a Schrödinger problem can be solved then there always existis a corresponding Dirac problem eq. (3.2) which can be exactly solved with ϕ identified as the superpotenial.

Zero Energy Solutions: We shall see in the next section that the zero energy solution play a crucial role in the phenomena of fermion number fractionization. Clearly eq. (3.5a) and (3.5b) have acceptable zero modes if the solution of the equations.

$$A\Psi_1^{(0)}(x) = 0 \qquad (3.9a)$$

$$A^+\Psi_2^{(0)}(x) = 0 \qquad (3.9a)$$

is square integrable. Using eq. (3.7) we see that the solutions of (3.9a) and (3.9b) are

$$\Psi_1^{(0)}(x) = N\begin{pmatrix} e^{-\int^x} \lambda\phi(y)d\,y \\ 0 \end{pmatrix} \qquad (3.10a)$$

$$\Psi_2^{(0)}(x) = N\begin{pmatrix} 0 \\ e^{-\int^x} \lambda\phi(y)d\,y \\ 0 \end{pmatrix} \qquad (3.10b)$$

Clearly $\Psi_1^{(0)}(x)\,\Psi_2^{(0)}(x)$ is square integrable if $\phi(\infty) > 0(\phi(\infty) < 0)$ and if $\phi(-\infty)$ has opposite sign to that $\phi(\infty)$. Thus we see that both H_+ cannot simultaneously have a zero mode. As an illustration, when $\phi(x)$ is the kink solution $\frac{\mu}{\sqrt{\lambda}}$ tanh $\mu\,x$ then it follows that only H has a zero mode with

$$\Psi_1^{(0)}(x) = N\begin{pmatrix} (\operatorname{sech}\mu\,x)^{\sqrt{\lambda}} \\ 0 \end{pmatrix} \qquad (3.11)$$

It is worth noting that this solution is self charge conjugate i.e.

$$\Psi_0^{\mathcal{L}}(x) = \sigma_3\Psi_0^*(x) = \Psi^{(0)}(x) \qquad (3.12)$$

This is also true for the general solutions (3.10a) and (3.10b) with eigenvalues +1 and -1 respectively.

4.Fermion Number Fractionization

We shall now discuss the quantization of the Dirac field in the solitonic background. To appreciate the role of the zero modes it might be worthwhile to first discuss the standard quantization around the normal vacuum which in our case is $\phi = \pm\mu\sqrt{\lambda}$. On expanding the Dirac field

$$\Psi(x,t) = \sum_k \left[e^{-i\varepsilon_k l} b_k u_k^{(+)} + e^{-i\varepsilon_k l} d_k^+ v_k^{(-)}(x)\right] \qquad (4.1)$$

Where $b_k^+(b_k)$ are the creation (annihilation) operators for the particles while d_k^+ (d_k) are the same for antiparticles. We can then immediate calculate the expression for the fermion number operator Q defined by

$$Q = \frac{1}{2}\int d\,x \sum_{i=1}^{2}[\Psi_i^+(x)\Psi_i(x) - \Psi_i(x)\Psi_i^+(x)] \qquad 4.2$$

by using eq. (4.1). we get

$$Q = \sum_k b_k^+\, b_k - d_k^+\, d_k \qquad (4.3)$$

Which immediately shows that the normal vacuum state $\phi = \pm\,\mu/\sqrt{\lambda}$ has zero fermion number: Q1 vacuum >=0. How does the discussion change in the case of solutions? In this case we have seen that the solution of Dirac equation in the background of soliton consists of a zero energy state and then non-zero energy Dirac spectrum which is symmetric about *E=0*. This is a consequence of the fact that the Dirac Lagrangian (3.1) is invariant under charge conjugation. As a result we find that the discussion now is almost parallel to the previous case except for the zero modes which however make a profound difference. The point is that now the expansion on of Dirac field (4.1) gets

$$\Psi(x) = a\Psi_0(x) + \sum_k \left[e^{-ie_k l} B_k U_k^{(+)}(x)\right] \qquad (4.4)$$

Where

$$+e^{-ie_k l} D_k^+\, V_k^{(-)}(x)$$

$$\{\Psi(x), \Psi^+(x')\}_{x_0=x_0'} = \delta(x-x') \qquad (4.5)$$

leads to

$$\{a, a^+\} = 1 \qquad (4.6)$$

As a result, the expression for Q defined y (4.2) takes the form

$$Q = \left(a^+a - \frac{1}{2}\right) + \sum_k (B_k^+\, B_k - D_k^+ D_k) \qquad (4.7)$$

This clearly shows that the soliton state is doubly degenerate having fermion number $\pm 1/2$. Let us denote the two degenerate states by $1\ S, +$and $1S, -\rangle$. Clearly

$$a \,|S,-\rangle =; a^{+} \,|S,+\rangle\, 0 \quad (4.8)$$

$$a \,|S,+\rangle = |S,-\rangle; a^{+}|S,-\rangle = |S,+\rangle \quad (4.9)$$

So that

$$Q|S,-\rangle = \frac{1}{2}|S,-\rangle; Q|S,+\rangle = \frac{1}{2}|S,+\rangle \quad (4.10)$$

Is this fractional fermion number allowed in quantum field theory? Answer is yes. The point is that in field theory the local operators must carry integral fermion number and hence the difference in fermion number between any two quantum states in a given sector must be integral:

$$N - N' - n, |n| = 0, 1, 2 \ldots.. \quad (411)$$

If the theory has charge conjugation symmetry then if a state has fermion number N then the antistate must have fermion

number $-N$ so that $2N = n$ $i.e.$, a state can have either an integer or half integer fermion number. It must be emphasized here that if the theory does not have charge conjugation symmetry then the state can have any fractional charge. Goldstone-Wilczek[14] have explicitly demonstrated this phenomenon in a model without charge conjugation symmetry.

This phenomena of fermion number of fractionization is not restricted to this model alone but occurs whenever solutions of Dirac equation in the topologically nontrival background are considered[7]. This phenomenon s at the heart of the so called Callan-Rubakov effect[15].

5.Experimental Relevance

It is well known that in polymers like polyacetylene $(CH)_x$ electrons move primarily in one-dimension.

Fig. 1

It has been observed that the polyacetylene appears in two phases A and B which are reflected images of each other. It has also been observed that many a times (CH), chain is in the A phase at one end while at the other end it is observed in B-phase. Clearly in between, at some point a transformation must occur from phase A to B

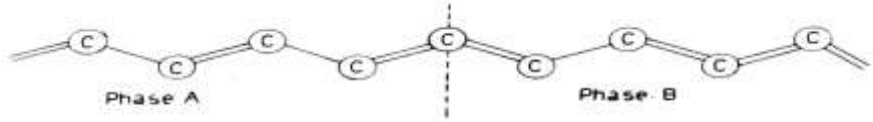

Fig. 2

The transformation between the two phases is what is termed as soliton. However, in all this analysis there is an extra complication arising from the doubling of the degrees of freedom due to spin. As a result one does not observe the charge fractionization but one does observe unusual charge spin relationship. One finds that whereas the charged solitons are spinless, the neutral ones carry spin-1/2. These have been experimentally observed in electron spin resonance experiments.

6.General Remarks and Open Problems

There are several situations in which one considers the solution of Dirac equation. One of the famous situation is when electron is moving in a plane under the influence of the external magnetic field. The Dirac equation reads

$$i\,\gamma^{\mu}\left(\partial_{\mu} - i\,eA_{\mu}\right)\Psi = 0 \qquad (6.1)$$

on choosing

$$\Psi(\vec{x},t) = e^{\iota\,E\,t}\Psi(\vec{x});\, A_0(\vec{x},t) = 0 \qquad (6.2)$$

We find that the problem to be solved is

$$H\Psi = -\vec{\alpha}.(\vec{p} - e\vec{A}) = E\Psi \qquad (6.3)$$

We choose

$$\alpha^1 = -\sigma^2, \alpha^2 = \sigma^1, \beta = \sigma^3 \qquad (6.4)$$

Note that

$$\{H,\beta\}_+ = 0 \qquad (6.5)$$

As a result of which σ^3 takes the positive energy eigen function to its charge conjugate negative energy are

$$\sigma^3\Psi_{\mathrm{E}} = \Psi_{-\mathrm{E}} \qquad (6.6)$$

We are interested in looking for the zero modes. To that purpose let $\Psi_0 = \begin{pmatrix} u \\ v \end{pmatrix}$ and choose the Coulomb gauge for A, which is assumed to be single valued and well behaved at the origin[16]

$$A^i = \varepsilon^{ij}\partial_j a \qquad (6.7a)$$

$$B = \varepsilon_{ij}\partial^j A^j = \vec{\nabla}^2 a \qquad (6.7b)$$

In this case the Dirac eq. (6.3) reduces to the pair

$$(\partial_x + i\,\partial y)u - e\left(\partial_x + i\,\partial_y\right)au = 0 \qquad (6.8a)$$

$$\left(\partial_x + i\,\partial_y\right)v - e\left(\partial_x + i\,\partial_y\right)a\,v = 0 \qquad (6.8b)$$

These eqs. have the solutions

$$u = \exp(e\ a)\ f\ (x + i\ y) \qquad (6.9a)$$

$$v = \exp(-e\ a)\ g\ (x - i\ y) \qquad (6.9a)$$

Where f and g are arbitrary entire functions. One can show that either of the self-conjugate solutions $\begin{pmatrix} u \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ v \end{pmatrix}$ are normalizable and that if

$$\phi = \int B\ d^2x \qquad (6.10)$$

then the no. of zero energy states on the largest integer less than $e\ \phi - 1$. This is a variant of the famous Antiyash-Singer Index theorem.

One of the most important case is the solution of the Dirac eq. (1.2). In several special cases such solution have already been obtained. For example

$$(i) A_\mu(\vec{x}) = V(\vec{x})\delta_{\mu 0}$$

This is the famous case of static potential. The most famous example is $V\ (x) = -e^2/|\vec{x}|$ which is exactly solvable. However, the classification of solutions of Dirac eq. for arbitrary *V(r)* in 1 or 3 dimensions is nonexistent

$$(ii) A_\mu(x) = A_l(x); A_o(x) = 0$$

Using tricks of super -symmetric quantum mechanics. Several solvable cases have been indenfied[4].

What about the zero modes of the Dirac equation in the background of charged vortices[8]. Only one mode has been obtained so far and that too for odd integer flux. Clearly is tisvery interesting to see if there are more zero modes and if there is an index theorem. Similarly it is worthwhile considering the zero modes of the σ-model with fermions which could be relevant in the context of high-T_c super-conductivity.

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