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Elzaki Transform Approach to Solve the Advection-Diffusion Equation: A Novel Perspective

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Abstract: The advection-diffusion equation plays a pivotal role in modeling transport phenomena in diverse scientific and engineering disciplines, encompassing fluid dynamics, environmental science, and heat transfer. This study explores a novel analytical approach to solving the advection-diffusion equation using the Elzaki transform, a robust integral transform method known for its efficiency in handling linear partial differential equations. By leveraging the Elzaki transform, the solution process is significantly simplified, transforming the governing equation into an algebraic form, which facilitates the derivation of exact solutions. The method is applied to specific initial and boundary conditions, and the resulting analytical solutions are validated through illustrative examples. Furthermore, the behavior of the solution over time and space is visualized using graphical representations, highlighting the efficiency of the Elzaki transform in addressing complex transport problems. This work demonstrates the potential of the Elzaki transform as a powerful tool for solving advection-diffusion equations, offering insights into its application for real-world scenarios.

Keywords: Advection-diffusion equation, Elzaki transform, analytical solutions, integral transform, transport phenomena, fluid dynamics, heat transfer, boundary conditions, initial conditions, transport modeling

I. Introduction

The advection-diffusion equation is a fundamental partial differential equation (PDE) that describes the combined effects of advection (transport due to a flow field) and diffusion (spreading due to concentration gradients) in various physical, chemical, and biological systems. This equation underpins numerous real-world phenomena, including pollutant dispersion in air and water, heat transfer in materials, and solute transport in porous media. Analytical solutions to the advection-diffusion equation provide deep insights into these processes and serve as benchmarks for validating numerical methods. However, finding exact solutions can be challenging, especially for complex boundary and initial conditions. In recent years, integral transform methods have emerged as powerful analytical tools for solving PDEs. Among these, the Elzaki transform has gained attention for its simplicity and efficiency in reducing differential equations to algebraic forms. The Elzaki transform extends the capabilities of classical transforms, such as the Laplace and Fourier transforms, by offering unique operational properties that simplify the solution process for linear and certain nonlinear differential equations. Its flexibility and robustness make it particularly suitable for problems involving transport phenomena. This study focuses on applying the Elzaki transform to the advection-diffusion equation, showcasing its ability to derive exact solutions for specific initial and boundary conditions. By transforming the PDE into a more manageable algebraic equation, the Elzaki transform simplifies the mathematical complexity, enabling a systematic and efficient solution process. The analytical solutions obtained are further analyzed and visualized to highlight the physical behavior of the system over time and space. This work aims to provide a novel perspective on solving the advection-diffusion equation, demonstrating the utility of the Elzaki transform in advancing the understanding and application of transport phenomena in diverse scientific and engineering contexts.

Elzaki and Kim (2015) demonstrated the capability of the method to handle nonlinear equations efficiently, even with complex boundary conditions. Their approach provided exact or approximate solutions that converged rapidly, highlighting the practical advantages of the method for engineering and physical applications. This work laid a foundation for further applications of the Elzaki transform in addressing diffusion and wave propagation problems. **Neamaty et al. (2016)** bridged the gap between fractional calculus and integral transforms, enabling precise modeling of physical phenomena characterized by memory and hereditary properties. The proposed transform was shown to simplify the mathematical complexity of fractional equations, offering an alternative to traditional numerical methods while maintaining accuracy and computational efficiency. **El-Tantawy et al. (2017)** focused on the formation, stability, and propagation of breather structures under varying plasma parameters. By employing advanced mathematical techniques, they revealed the critical roles of ion density, electron temperature, and plasma frequency in determining the characteristics of breather waves. This work provided new insights into nonlinear wave phenomena in electronegative plasma systems. **El-Tantawy and Wazwaz (2018)** extended the understanding of plasma systems with non-equilibrium particle distributions. The study demonstrated the relevance of the mKdV equation in capturing intricate nonlinear interactions in dusty plasmas. **Nadeem et al. (2019)** proposed a innovative approach showcased the versatility of combining Laplace transforms with iterative techniques to address complex boundary and initial conditions. The method was validated through several examples, demonstrating its efficiency and potential for high-order nonlinear problems. **Anjum and He (2019)** introduced a streamlined application of the variational iteration method using Laplace transforms. They simplified the computational burden associated with iterative methods by leveraging the integral properties of the Laplace transform. Their approach was particularly effective for nonlinear differential equations, making the method more accessible for applied mathematics and engineering problems. **Ul Rahman et al. (2019)** applied the He-Elzaki method to model the spatial diffusion of biological populations. Their study emphasized the adaptability of the method for biological systems, capturing population dynamics influenced by spatial diffusion and other environmental factors. The work highlighted the potential of the He-Elzaki transform for addressing ecological and biological diffusion problems. **El-Tantawy et al. (2021)** explored the effects of nonlinearity, dispersion, and ion composition on wave dynamics, providing valuable insights into ionospheric plasma behavior. The proposed methods proved effective for capturing the complexities of nonplanar wave structures in realistic plasma environments. **Aljahdaly et al. (2022)** achieved high accuracy and stability, making it a robust tool for simulating solitonic phenomena in dissipative systems. The study highlighted the importance of time-stepping schemes in accurately resolving solitonic behaviors in plasma physics. **El-Tantawy et al. (2022)** emphasized the interplay between nonlinearity, dispersion, and cylindrical geometry in shaping wave dynamics. By offering a comprehensive analysis of rogue wave behavior, this study enriched the understanding of nonlinear phenomena in complex plasma systems. **El-Tantawy et al. (2022)** highlighted the impact of nonplanar geometries and complex plasma properties on wave dynamics, offering new theoretical tools for plasma modeling. **Nadeem et al. (2023)** demonstrated the effectiveness of fractional derivatives in modeling anomalous diffusion processes across multiple dimensions. The study underscored the importance of fractional calculus in capturing complex physical behaviors that deviate from classical diffusion models. **Ji-Huan et al. (2023)** examined the limitations and future prospects of traditional transforms, such as Laplace and Fourier, for solving modern mathematical problems. Their work proposed extensions and alternatives to classical techniques, addressing the growing demand for advanced analytical tools in nonlinear and fractional systems. This study provided a comprehensive overview of the challenges and opportunities in the field of integral transforms. **El-Tantawy et al. (2024)** demonstrated the method's ability to accurately describe shock wave dynamics in fractional systems, highlighting its potential for solving nonlinear partial differential equations in various fields. **et al. (2024)** contributed to the understanding of coupled nonlinear systems, particularly in contexts where fractional derivatives play a critical role. This work expanded the applicability of fractional models to a broader range of physical and engineering problems.

II. ADVECTION DIFFUSION EQUATION

The **advection-diffusion equation** is a widely used partial differential equation (PDE) in various fields, such as fluid mechanics, environmental science, and chemical engineering. Its general form is:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} \quad (1)$$

where:

$C(x,t)$: the concentration of the substance at position x and time t

u: the velocity of advection (constant or variable), D: the diffusion coefficient (constant or variable),

t: time,

x: spatial coordinate.

(i) **Initial Condition**: $C(x,0) = \sin\left(\frac{\pi x}{L}\right)$ (2)

(ii) **Dirichlet Boundary Conditions**:

$$C(0,t) = C(L,t) = 0 \tag{3}$$

III. Definition of Elzaki Transform

The Elzaki transform of a function $C(x,t)$ is defined as:

$$E[C(x,t)] = \bar{C}(x,s) = \int_0^\infty e^{-\frac{t}{s}}\ C(x,t)dt \tag{4}$$

IV. Implementation of Elzaki transform to the solution of proposed PDE

Taking the Elzaki transform of the given PDE (1) with respect to t, we get:

$$E\left[\frac{\partial C}{\partial t}\right] + u\left[\frac{\partial C}{\partial x}\right] = DE\left[\frac{\partial^2 C}{\partial x^2}\right] \tag{5}$$

$$s\bar{C}(x,s) - \frac{1}{s}C(x,0) + u\frac{\partial \bar{C}(x,s)}{\partial x} = D\frac{\partial^2 C}{\partial x^2}\bar{C}(x,s)$$

$$D\frac{\partial^2 \bar{C}}{\partial x^2} - u\frac{\partial \bar{C}}{\partial x} - s\bar{C} = -\frac{1}{s}\sin\left(\frac{\pi x}{L}\right) \tag{6}$$

The above equation is a **second-order linear ODE** in x. Its homogeneous part is**:**

Auxiliary equation is $Dm^2 - um - s = 0$

$$m = \frac{u \pm \sqrt{u^2+4Ds}}{2D} = \frac{u+\sqrt{u^2+4Ds}}{2D}, \frac{u-\sqrt{u^2+4Ds}}{2D}\ (m_1, m_2) \text{ say}$$

$$\text{Complementary function} = Ae^{m_1 x} + Be^{m_2 x} \tag{7}$$

The particular integral is $\frac{\sin\left(\frac{\pi x}{L}\right)}{s\left[s+D\frac{\pi^2}{L^2}\right]}$

The complete solution in the Elzaki domain is:

$$\bar{C}(x,s) = Ae^{m_1 x} + Be^{m_2 x} + \frac{\sin\left(\frac{\pi x}{L}\right)}{s\left[s+D\frac{\pi^2}{L^2}\right]} \tag{8}$$

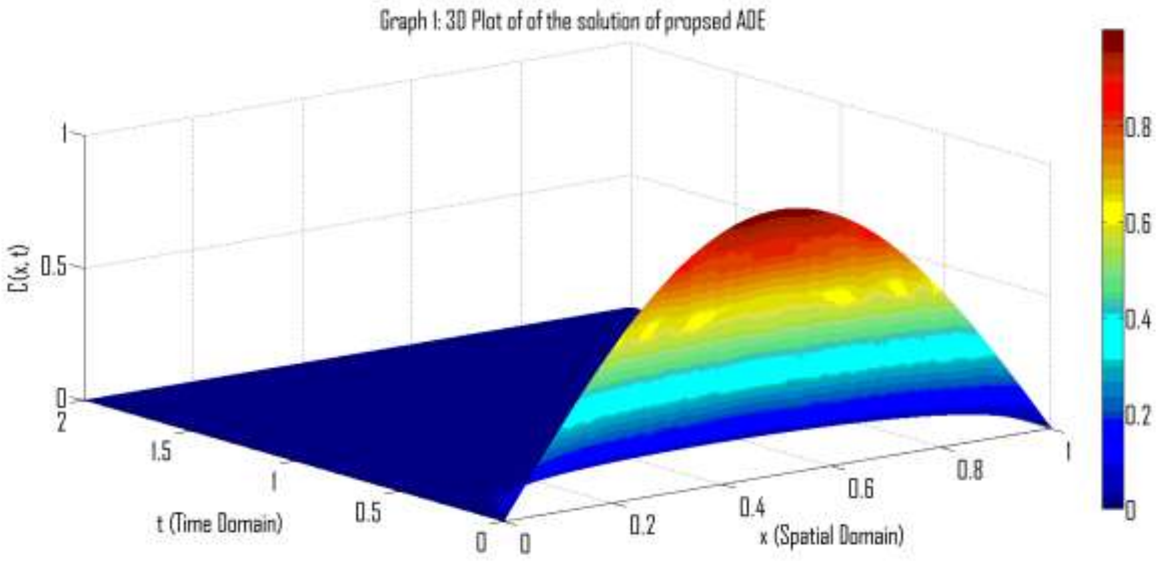
Using boundary condition (3) in equation (8), we get

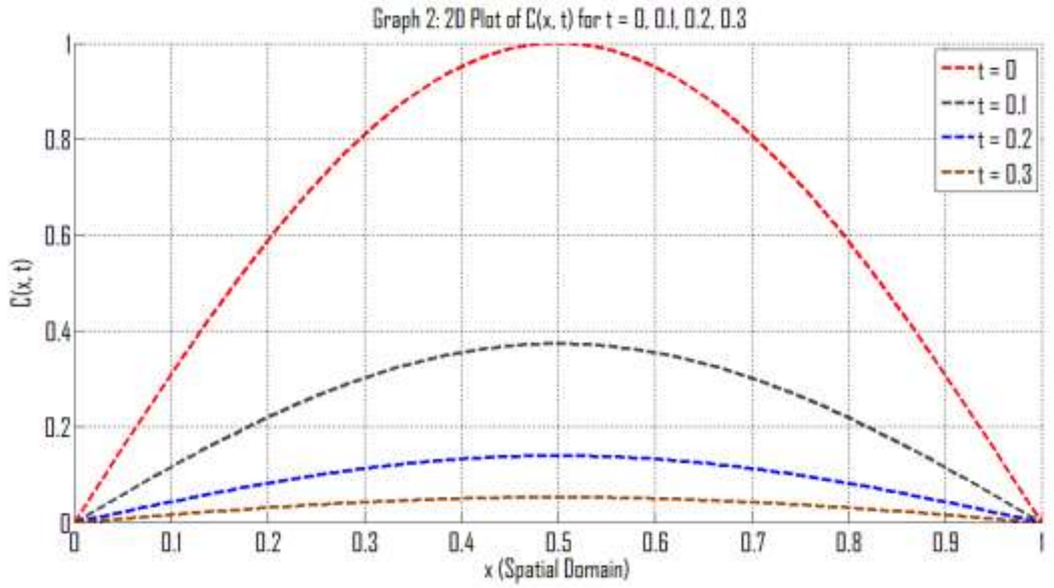
$$\bar{C}(x,s) = \frac{\sin\left(\frac{\pi x}{L}\right)}{s\left[s + D\frac{\pi^2}{L^2}\right]} \tag{9}$$

Finally, take the inverse Elzaki transform to find $C(x,t)$:

$$C(x,t) = e^{-\frac{D\pi^2 t}{L^2}} \sin\left(\frac{\pi x}{L}\right) \tag{10}$$

V. Results and Discussion





The 3D plot in graph (1) depicts the solution of a proposed Advection-Diffusion Equation (ADE) over a time-space domain. The horizontal axes represent the spatial domain (x) and the temporal domain (t), while the vertical axis corresponds to the solution $C(x,t)$, which is likely a concentration profile. The color gradient indicates the magnitude of $C(x,t)$, with blue representing the lowest values and red the highest. The solution demonstrates an increase in $C(x,t)$ over both space and time, peaking near the midpoint of the spatial domain and then gradually decreasing. This pattern may signify a diffusion-dominated process with a source at x and a time-dependent decay or dispersion. The graph highlights the dynamic interplay between advection and diffusion, with clear temporal and spatial gradients in the concentration distribution.

The 2D plot in graph (2) shows the solution $C(x,t)$ of the advection-diffusion equation at different time steps $(t = 0,0.1,0.2,0.3)$ over the spatial domain (x) from 0 to 1. The vertical axis represents $C(x,t)$, while the horizontal axis represents the spatial domain. Each curve corresponds to a specific time step, with color-coded and dashed lines indicating the progression of the solution over time. At $t = 0$ (red curve), the solution peaks symmetrically in the middle of the domain, representing the initial concentration distribution. As time progresses $(t = 0.1,0.2,0.3)$, the concentration decreases in magnitude, spreads out, and flattens, as seen in the black, blue, and brown curves. This behavior suggests the effects of diffusion causing the concentration to disperse and diminish over time. The symmetry in the spatial domain indicates no directional advection, emphasizing the dominant role of diffusion in this process.

Table 1: Numerical Values of C(x,t) for Various Spatial (x) and Temporal (t) Points

t \ x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0	0	0.309017	0.587785	0.809017	0.951057	1	0.951057	0.809017	0.587785
0.1	0	0.279975	0.532544	0.732984	0.861674	0.906018	0.861674	0.732984	0.532544
0.2	0	0.253662	0.482495	0.664097	0.780693	0.820869	0.780693	0.664097	0.482495
0.3	0	0.229823	0.437149	0.601684	0.707322	0.743722	0.707322	0.601684	0.437149
0.4	0	0.208224	0.396065	0.545136	0.640846	0.673825	0.640846	0.545136	0.396065
0.5	0	0.188654	0.358842	0.493903	0.580618	0.610498	0.580618	0.493903	0.358842

VI. Results and Discussion

The Elzaki transform has proven to be a powerful and efficient tool for solving the advection-diffusion equation, offering a novel perspective on analytical approaches to transport phenomena. By reducing the complexity of the partial differential equation to an algebraic form, the Elzaki transform facilitates the derivation of exact solutions under specific initial and boundary conditions. This study has demonstrated how the method not only simplifies the mathematical process but also provides insights into the dynamic behavior of advection and diffusion over time and space. The analytical solutions derived using the Elzaki transform are validated through graphical representations, which highlight the concentration profiles and their evolution under varying conditions. These solutions serve as benchmarks for understanding real-world phenomena such as pollutant dispersion, heat transfer, and solute transport in porous media. Moreover, the Elzaki transform's versatility and ease of application make it a promising tool for extending the analysis to more complex scenarios, including nonlinearities, variable coefficients, and multi-dimensional problems. In conclusion, this work underscores the transformative potential of the Elzaki transform in advancing analytical methods for solving advection-diffusion equations. It bridges the gap between theoretical exploration and practical application, paving the way for further research and innovation in the field of transport modeling and analysis.

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