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A Generalization of Inverse Rayleigh Distribution

Sankaran K K

Associate Professor
Department of Statistics
Sree Narayana College, Nattika

Abstract : In this article, a new four parameter lifetime model called Generalized Inverse Rayleigh (GIR) distribution is defined and studied. This distribution derived from Kumaraswamy inverse Marshall-Olkin-G family of distribution. Inverse Rayleigh distribution has many applications in lifetime studies. Various properties of the new model are discussed including closed forms expressions for moments and quantiles. The maximum likelihood method the model parameters. The proposed distribution is fitted to a real data set and it is shown that the distribution is more appropriate for modeling in comparison with some other competitive models.

***Index Terms* - Inverse Rayleigh distribution, Kumaraswamy distribution, Marshall-Olkin family of distributions, Maximum likelihood, Moments.**

I. Introduction

Modelling and analysis of lifetime phenomena are important aspects of statistical work in a wide variety of scientific and technological field. The field of lifetime data analysis has grown and expanded rapidly with respect to methodology, theory, and fields of applications. In the context of modelling the real-life phenomena, continuous probability distributions and many generalizations or transformation methods have been proposed. These generalizations, obtained either by adding one or more shape parameters or by changing the functional form of the distribution, make the models more sufficient for many applications.

In the last two decades researchers have greater intention toward the inversion of univariate probability models and their applicability under inverse transformation. The inverse distribution is the distribution of the reciprocal of a random variable. Dubey (1970) proposed inverted beta distribution, Voda (1972) studied inverse Rayleigh distribution, Folks and Chhikara (1978) proposed inverse Gaussian distribution, Prakash (2012) studied the inverted exponential model, Sharma et al. (2015) introduced inverse Lindley distribution, Gharib et al. (2017) studied Marshall-Olkin extended inverse Pareto distribution, Al-Fattah et al. (2017) introduced inverted Kumaraswamy distribution and Rana and Muhammad (2018) introduced Marshall-Olin extended inverted Kumaraswamy distribution.

The inverse Rayleigh (IR) distribution is commonly used in statistical analysis of lifetime or response time data from reliability experiments. For the situations in which empirical studies indicate that the hazard function might be unimodal, the IR distribution would be an appropriate model. Initially, Treyer (1964) introduced the inverse Rayleigh distribution as a model for analyzing reliability and survival data. The model later underwent further examination by Voda (1972), who observed that the lifetime distributions of various experimental units could be closely approximated with the inverse Rayleigh distribution. Additionally, Voda (1972) explored its properties and provided a maximum likelihood (ML) estimator for the scale parameter. Gharraph(1993) conducted an in-depth analysis of the inverse Rayleigh distribution, deriving five key measures of location: the mean, harmonic mean, geometric mean, mode, and median. Furthermore, Gharraph explored various estimation methods to determine the unknown parameter of this distribution. A numerical comparison of these estimation techniques was conducted, focusing on their bias and root-mean-squared error (RMSE), providing valuable insights into their performance and applicability. Almarashi et al. (2020) propose a two-parameter extension of the inverse Rayleigh distribution, employing the half-logistic transformation to address limitations in modeling moderately right-skewed or near-symmetrical lifetime data. Their theoretical contributions encompass mathematical properties and empirical evidence, demonstrating the model's effectiveness in handling diverse right-skewed datasets. Babokan and Al-Shehri (2021) introduced a new generalized inverse Rayleigh distribution with applications in five data set in different field. Furthermore, Chiodo et al. (2022) introduced the compound inverse Rayleigh distribution as a model tailored for extreme wind speeds, essential in wind power generation and turbine safety evaluation. They provide a practical framework for real-world data analysis, accompanied by a novel Bayesian estimation approach, supported by extensive numerical simulations and robustness assessments. Also Shala and Merovci (2024) developed a new three-parameter inverse Rayleigh distribution using generalized transmuted family of distributions.

The generation of new distributions by adding one or more parameters to standard distributions enhances their applicability to complex data across various fields. Motivated by this approach, several authors have proposed different methods for generating new distributions. These include the Marshall-Olkin-G distribution (1997), the Beta-G distribution introduced by Eugene et al. (2002), the Kumaraswamy-G distribution by Cordeiro and Castro (2011), and the McDonald-G distribution by Alexander et al. (2012). Shaw and Buckley (2009) introduced the transmuted-G class of distributions, which was further expanded by the development of the exponentiated transmuted-G distribution (2017) and the generalized transmuted G distribution (2017). Marshall and Olkin(1997) introduced a new family of distributions by adding a parameter to a family of distributions. They started with a parent survival function $\bar{F}(x)$ and considered a family of survival functions given by

$$\overline{G}(x;p) = \frac{p\overline{F}(x)}{F(x) + p\overline{F}(x)}, \quad p > 0 \quad x \in \mathbb{R}. \tag{1}$$

They described the motivation for the family of distributions (1) as follows:
Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with survival function $\bar{F}(x)$. Let

$$U_N = \min(X_1, X_2, \ldots, X_N), \tag{2}$$

where N is the geometric random variable with probability mass function (pmf) $P(N = n) = p(1-p)^{n-1}$, for n = 1,2,. . and $0 < p < 1$ and independent of X_0s Then the random variable U_N has the survival function given by (1). If $p > 1$ and N is a geometric random variable with pmf of the form $P(N = n) = \frac{1}{p}(1-\frac{1}{p})^{n-1}$, then the random variable $V_N = \max(X_1, X_2, \ldots, X_N)$ also has the survival function as (1).

If $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with distribution in the family (1), and if N has a geometric distribution on {1,2, . . . } then $\min(X_1, \ldots, X_N)$ and $\max(X_1, \ldots X_N)$ have distributions in the family. The extreme value distributions are limiting distribution for extreme, and as such they are sometimes useful approximations. In practice, a random variable of interest may be the extreme of only finite, possibly random, number N of random variables. When N has a geometric distribution, the random variable has a particular nice stability property, not unlike that of extreme value distributions. The geometric-extreme stability property of G(x; p) is rather remarkable, and it depends upon the fact that a geometric sum of i.i.d. geometric random variables has a geometric distribution. This partially explains why random minimum stability cannot be expected if the geometric distribution is replaced by some other distribution on {1,2, . . }. For more discussion on geometric-extreme stability, see Arnold(1986) and Marshall and Olkin (1997).

Heavy-tailed models have been used in a variety of fields, such as mathematical finance, financial economics and statistical physics. In the framework of integer valued distributions, the discrete stable is a well known heavy-tailed law originally suggested by Steutel(1979). Jayakumar(2003) generalized the concept of Poisson mixtures to discrete stable mixtures and showed that, the distributions on Z_+ that can be approximated by mixtures of discrete Linnik distributions are discrete stable mixtures. Christoph (1998) emphasized that the discrete stable law may be seen as a special case of discrete Linnik law studied in Devroye(1993). Hence owing to the extra parameter, the discrete Linnik is a heavy-tailed distribution family which is more exible than the discrete stable. Discrete Linnik distribution is a rich family of distributions which includes many important distributions. It belongs to the class of discrete self decomposable distributions.

The pgf of discrete Linnik distribution with parameters α, c and υ is

$$H(s) = \begin{cases} \left(\frac{1}{1+c(1-s)^{\alpha}}\right)^{\nu} & \text{for} \quad 0 < \nu < \infty \\ e^{-c(1-s)^{\alpha}} & \text{for} \quad \nu = \infty. \end{cases}$$

Jayakumar and Sankaran (2019) introduced a new family of distributions with parameters α, c and υ having survival function

$$\bar{G}(x,\alpha,c,\nu) = \frac{(1+c)^{\nu} - [1+cF^{\alpha}(x)]^{\nu}}{[(1+c)^{\nu}-1][1+cF^{\alpha}(x)]^{\nu}}. \qquad (3)$$

Note that the survival function in (3) is the survival function of U_N in (2), where X_i's are i.i.d. random variables with cdf F(x) and N is truncated discrete Linnik distribution with parameters α, c and υ and N is independent of X_i's. It can be seen that the family of distributions generated through truncated negative binomial and truncated discrete Linnik are not extreme stable.

Kumaraswamy(1980) introduced a probability distribution for handling double bounded random processes with varied hydrological applications. The cumulative distribution function (cdf) of Kumaraswamy distribution is given by

$$F(x) = 1 - \{1 - x^{a}\}^{b}; \quad a > 0,\ b > 0,\ x \in [0,1]. \qquad (4)$$

The beta distribution also provides the premier family of continuous distribution on bounded support which has been utilized extensively in statistical theory and practice (see Nadarajah (2007)). Gupta(2004) provides a comprehensive account of the theory and applications of beta family of distributions. Like beta distribution, Kumaraswamy distribution also originally de_ned on the unit interval [0, 1] but easily extended to any finite interval and can take an amazingly great variety of forms. Thus it can be fitted practically to any data representing a phenomenon in almost any field of applications. One interpretation for integer-valued a and b through maxima and minima of i.i.d. random components is by Jones(2009). If we assuming that a = m and b = n are positive integers, we can find, x^m is the cdf of the maximum of i.i.d. standard uniform variables, with the corresponding survival function 1 - x^m . Thus, the quantity (1- x^m }^{n} in (4) is the minimum of n such random variables, with G being the corresponding cdf. This property discussed in Jones(2009), motivated the name minimax for this distribution. Kozubowski(2018) extended this interpretation to the general Kumaraswamy distribution using the result of min/max of i.i.d. components with random number to the relevant pgf.

This paper is organized as follows. We introduce Kumaraswamy inverse discrete Linnik G (Kw-IDL-G) family of distributions in Section 2 and discuss its various sub models. In Section 3, a sub model of Kw-IDL-G, namely, Kumaraswamy inverse Marshall-Olkin family is obtained. As a special case, Kumaraswamy inverse Marshall-Olkin Rayleigh distribution, a new generalization of inverse Rayleigh (GIR) distribution is studied in detail. It can be seen that GIR distribution contains Kumaraswamy inverse Marshall-Olkin exponential distribution, Kumaraswamy inverse generalized exponential distribution, Kumaraswamy inverse exponential distribution, Marshall- Olkin generalized exponential distribution, Marshall-Olkin exponential distribution, generalized exponential distribution and exponential distribution. In Section 4, some structural properties of GIR distribution such as moments, quantiles and mean residual function are studied. Estimation of the modal parameters by maximum likelihood is performed in Section 5. An application to a real data set to illustrate the potentiality of the new family is presented in Section 6. It can be seen that GIR distribution performs well compared to several well known distributions. The paper is concluded in Section 7.

II. KUMARASWAMY INVERSE DISCRETE LINNIK G FAMILY OF DISTRIBUTIONS

Let X follows truncated discrete Linnik family of distributions with survival function S(.) and baseline distribution function F(.). Then $Y = \frac{1}{X}$ is an inverse truncated discrete Linnik random variable with cumulative distribution function (cdf) G(x) given by

$$\begin{aligned} G_Y(x) &= P(Y \le x) \\ &= P(\frac{1}{X} \le x) \\ &= P(X \ge \frac{1}{x}) \\ &= S(1/x) \\ &= \frac{(1+c)^{\theta} - [1+cF^{\beta}(1/x)]^{\theta}}{[(1+c)^{\theta}-1][1+cF^{\beta}(1/x)]^{\theta}}, \quad \beta,\theta,c > 0;\ x > 0 \qquad (5) \end{aligned}$$

Hence, we obtain a new family of distributions, which we named as inverse family of distributions generated through discrete Linnik G distribution.

The probability density function (pdf) and the hazard rate function (hrf) of a random variable from the introduced family are respectively,

$$g(y,\beta,c,\theta)=\frac{\beta\theta c(1+c)^{\theta}y^{-2}f(1/y)F^{\beta-1}(1/y)}{[(1+c)^{\theta}-1][1+cF^{\beta}(1/y)]^{\theta+1}}, \tag{6}$$

and

$$h(y,\beta,c,\theta)=\frac{\beta\theta cF^{\beta-1}(1/y)f(1/y)}{[1+cF^{\beta}(1/y)][(1+cF^{\beta}(1/y))^{\theta}-1]}. \tag{7}$$

We define the cdf of Kumaraswamy inverse truncated discrete Linnik family of distributions as

$$F(y)=1-[1-G(y)^{a}]^{b} \tag{8}$$

For each baseline G, the Kumaraswamy inverse truncated discrete Linnik G cdf is given by (8). It can be seen that (8) provides a class of wider family of continuous distributions. It includes the Kumaraswamy inverse discrete Mittag-Leffler G family of distributions, Kumaraswmay inverse truncated negative binomial G family of distributions, Kumaraswamy inverse Marshall-Olkin G family of distributions, Kumaraswamy inverse G family of distributions etc.

III. KUMARASWAMY INVERSE MARSHALL-OLKIN RAYLEIGH DISTRIBUTION

For analytical tractability, let β = 1 and θ= 1, then the cdf of (5) reduces to inverse Marshall- Olkin G family of distributions as

$$G(y)=\frac{p\overline{F}(1/y)}{1-(1-p)\overline{F}(1/y)}.$$

Now, as a special case, let X follows Rayleigh distribution with parameter $\lambda > 0$ having cdf $F(x) = 1- e^{-(\lambda x)^2}$ and pdf $f(x) = 2\lambda^2 x e^{-(\lambda x)^2}$. Hence, the cdf of the random variable Y is given by

$$\begin{aligned} F(y;a,b,p,\lambda) &= 1-\left\{\left[1-\frac{pe^{-(\frac{\lambda}{y})^2}}{1-(1-p)e^{-(\frac{\lambda}{y})^2}}\right]^{a}\right\}^{b} \\ &= 1-\left\{\left[\frac{1-e^{-(\frac{\lambda}{y})^2}}{1-(1-p)e^{-(\frac{\lambda}{y})^2}}\right]^{a}\right\}^{b} \end{aligned} \tag{9}$$

We refer to this new distribution having cdf (9) as Kumaraswamy inverse Marshall-Olkin Rayleigh distribution and write it as GIR (y; a, b, p, λ).
The pdf of GIR is given by

$$f(y)=\frac{2abp\lambda^2y^{-3}e^{-a\left(\frac{\lambda}{y}\right)^2}\left(1-e^{-a\left(\frac{\lambda}{y}\right)^2}\right)}{\left[1-(1-p)e^{-a\left(\frac{\lambda}{y}\right)^2}\right]^{a+1}}\left\{\left[\frac{1-e^{-(\frac{\lambda}{y})^2}}{1-(1-p)e^{-(\frac{\lambda}{y})^2}}\right]^{a}\right\}^{b-1} \tag{10}$$

The GIR models very flexible distribution that approaches when its parameters are changed. The new distribution includes as special cases of well-known models namely Kumaraswamy inverse Rayleigh (KwIR), Kumaraswamy inverse exponential (KwIE), Kumaraswamy inverse Marshall- Olkin exponential(KwMOIE), Inverse Marshall-Olkin Rayleigh(IMOR), exponentiated inverse Rayleigh (EIR), exponentiated inverse exponential (EIE), generalized inverse Rayleigh (GIR), generalized inverse exponential (GIE),inverse Rayleigh (IR) and inverse exponential (IE) models.

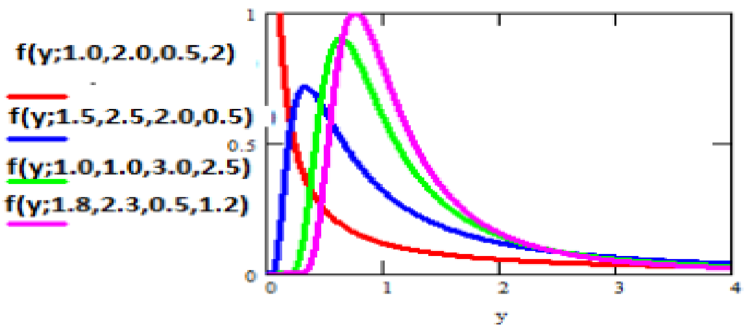

Figure 1: The plots of the pdf of GIR distribution.

The survival function S(y), hazard rate function h(y) and cumulative hazard rate function H(y) of Y are respectively, given by

$$S(y) = \left\{\left[\frac{1-e^{-(\frac{\lambda}{y})^2}}{1-(1-p)e^{-(\frac{\lambda}{y})^2}}\right]^a\right\}^b$$

$$h(y) = \frac{2abp\lambda^2 y^{-3} e^{-a(\frac{\lambda}{y})^2}\left[1-(1-p)e^{-(\frac{\lambda}{y})^2}\right]^{-(a+1)}}{\left[\frac{1-e^{-(\frac{\lambda}{y})^2}}{1-(1-p)e^{-(\frac{\lambda}{y})^2}}\right]^a}$$

$$H(y) = -b\log\left\{\left[\frac{1-e^{-(\frac{\lambda}{y})^2}}{1-(1-p)e^{-(\frac{\lambda}{y})^2}}\right]^a\right\}$$

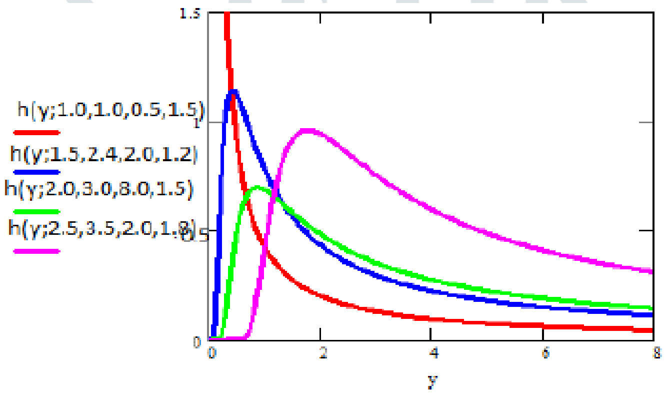

Figure 2: The plots of the hazard rate of GIR distribution

IV. SOME STATISTICAL PROPERTIES

IV.1 Linear representation

Consider a power series given by

$$(1-z)^{-\alpha} = \sum_{i=1}^{\infty}\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)j!}z^j, \quad |z|>1,\ \alpha>0 \tag{11}$$

Applying expansion (11) to equation (10) gives

$$f(y) = 2abp\lambda^2 y^{-3}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{j!\Gamma(b-1)}e^{\left[-a(j+1)(\frac{\lambda}{y})^2\right]}\left[1-(1-p)e^{\left[-(\frac{\lambda}{y})^2\right]}\right]^{-a(j+1)-1} \tag{12}$$

Applying (11) again to (12), we obtain

$$f(y) = \sum_{j,i=0}^{\infty}\frac{(-1)^j\Gamma(b)\Gamma(aj+a+i+1)}{j!i!\Gamma(b-j)\Gamma(aj+a+1)}p^{-a(j+1)}\left(1-\frac{1}{p}\right)^i 2ab\lambda^2 y^{-3}e^{-(aj+a+1)(\frac{\lambda}{y})^2}$$

The last equation can be expressed as

$$f(y) = \sum_{j,i=0}^{\infty}\nu_{j,i}h_{aj+a+i}(y), \tag{13}$$

Where $h_{\alpha j+a+i}(y)$ is the inverse Rayleigh density with scale parameter $\lambda(\alpha j+a+i)^{1/2}$ and $\upsilon_{j,i}$ is a constant term given by

$$\nu_{j,i} = \frac{(-1)^j a\Gamma(b+1)\Gamma(aj+a+1)}{p^{a(j+1)} j! i! \Gamma(b-j)\Gamma(aj+a+1)} \left(1 - \frac{1}{p}\right)^i .$$

Therefore, the GIR density is given by a linear combination of the inverse Rayleigh densities. So, several structural properties can be derived from those of the inverse Rayleigh distribution.

Similarly, the cdf of Y in (9) admits a linear representation given by

$$F(y) = \sum_{j,i=0}^{\infty} \nu_{j,i} H_{aj+a+i}(y),$$

Where $H_{\alpha j+a+i}(y)$ is the cdf of the inverse Weibull with scale parameter $\lambda(\alpha j+a+i)^{1/\beta}$ and shape parameter 2.

IV.2 Moments

Let X be a random variable having inverse Rayleigh distribution with parameter λ. The first moment of X is given by $\mu_{1'} = \lambda\,\Gamma(1/2)$ and the

The r^{th} ordinary moment, say $\mu_{r'}$ of Y is given by

$$\mu'_r = E(Y^r) = \sum_{j,i=0}^{\infty} \nu_{j,i} \int_0^{\infty} y^r h_{aj+j+i}(y)dy$$

Thus we have,

$$\mu'_1 = \lambda\Gamma\left(\frac{1}{2}\right) \sum_{j,i=0}^{\infty} \nu_{j,i}(aj+a+i)^{\frac{1}{2}} \qquad (14)$$

Hence in equation (14), we obtain the mean of Y.

IV.3 Quantile function

The quantile function of Y is determined by inverting (9) as

$$Q(u) = \lambda \left\{ \log\left[1 + \frac{1-\left(1-(1-u)^{\frac{1}{b}}\right)^{\frac{1}{a}}}{p\left(1-(1-u)^{\frac{1}{b}}\right)^{\frac{1}{a}}}\right]\right\}^{-\frac{1}{2}}, \quad 0 < u < 1.$$

Simulating the GIR random variables is straightforward. If *U* is a uniform variate in the unit interval (0,1), the random variable *Y* = *Q(U)* follows GIR density.

The effects of the additional shape parameters *a* and *b* on the skewness and kurtosis of the new distribution can be based on quantile measures. The well-known Bowley's skewness and Moors' kurtosis are respectively, defined by

$$B = \frac{Q_{\frac{3}{4}} + Q_{\frac{1}{4}} - 2Q_{\frac{1}{2}}}{Q_{\frac{3}{4}} - Q_{\frac{1}{4}}} \quad \text{and} \quad M = \frac{Q_{\frac{3}{8}} - Q_{\frac{1}{8}} + Q_{\frac{7}{8}} - Q_{\frac{5}{8}}}{Q_{\frac{6}{8}} - Q_{\frac{2}{8}}}.$$

These measures are even-though less sensitive to outliers, but they exist even for distributions without moments.

IV.4 Moments of the Residual and Reversed Residual life

The n^{th} moment of the residual life, $m_n(t) = E[(Y - t)^n \mid Y>t]$, n = 1, 2, . . . uniquely determines F(y).

Using equation (13), we can write

$$m_n(t) = \frac{1}{S(t)} \sum_{j,i=0}^{\infty} \nu_{j,i} \int_t^{\infty} (x-t)^n h_{aj+a+i}(y)dy.$$

By using the binomial R expansion and the upper incomplete gamma function, we obtain

$$m_n(t) = \frac{1}{S(t)} \sum_{r=0}^{n} \frac{(-1)^{n-r} n! t^{n-r} \lambda^r}{r!(n-r)!(aj+a+i)^{-(\frac{r}{2})}} \sum_{j,i=0}^{\infty} \nu_{j,i} \Gamma\left(1-\frac{r}{2}, (aj+a+i)(\frac{\lambda}{t})^2\right).$$

Another function is the mean residual life (MRL) function at age t defined by $m_1(y) = E[(Y-y)|Y > y]$, which represents the expected additional life length for a unit which is alive at age y.
The MRL of Y follows by setting n = 1 in the last equation.
The n^{th} moment of the reversed residual life $M_n(t) = E[(t - Y)^n |Y \leq t]$ for t > 0; n = 1, 2, . . .uniquely determines F(y).
Using equation (13), we can write

$$M_n(t) = \frac{1}{F(t)} \sum_{j,i=0}^{\infty} \nu_{j,i} \int_0^{t} (t-y)^n h_{aj+a+i}(y)dy.$$

By using the binomial expansion and the lower incomplete gamma function, we obtain

$$M_n(t) = \frac{1}{F(t)} \sum_{r=0}^{n} \frac{(-1)^r n! t^{n-r}}{r!(n-r)!} \lambda^r (aj+a+i)^{\frac{r}{2}} \sum_{j,i=0}^{\infty} \nu_{j,i} \gamma\left(1-\frac{r}{2}, (aj+a+i)(\frac{\lambda}{t})^2\right).$$

The mean inactivity time (MIT), also called mean reversed residual life (MMRL) function, is defined by $M_1(t) = E[(t - Y) |Y \leq t]$ It represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in (0; y). The MIT of the GIR distribution can be determined by setting n = 1 in the last equation.

V. ESTIMATION OF THE PARAMETERS

Several approaches for parameter estimation were proposed in the statistical literature, but maximum likelihood method is the most commonly employed. The MLEs method enjoy desirable properties for constructing confidence intervals. The estimation of the parameters of the GIR distribution by maximum likelihood for complete data sets. Let $y = (y_1, y_2, \ldots, y_n)$ be a random sample of this distribution with unknown parameter vector $\theta = (a, b, p, \lambda)'$. Then, the log-likelihood function for θ, is given by

$$\log\ L = n\log(2abp\lambda^2) - 3\sum_{i=1}^{n} \log y_i - a\sum_{i=1}^{n}\left(\frac{\lambda}{y_i}\right)^2 + \sum_{i=1}^{n}\log(1-u_i)$$
$$-(a+1)\sum_{i=1}^{n}\log(v_i) + (b-1)\sum_{i=1}^{n}\log[1 - v^{-a}(1-u_i)^a]$$

where $u_i = 1 - e^{-(\frac{\lambda}{y_i})^2}$ and $v_i = 1-(1-p)(1-s_i)$.

The MLE of θ can be determined by maximizing log L (for a given y) either directly by using the Mathcad, R (nlm or optim function), SAS (PROC, NLMIXED) or by solving the nonlinear system obtained by differentiating this equation and equating its four components to zero.

Let $s_i = \left(\frac{\lambda}{y_i}\right)^2 (1-u_i)^a \log\left(\frac{\lambda}{y_i}\right)$ and $t_i = (1-u_i)\left(\frac{2}{\lambda}\right)\left(\frac{2}{y_i}\right)^2$.

The derivative of the log-likelihood function with respect to the parameters a, b, p, λ are given respectively by

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\left(\frac{\lambda}{y_i}\right)^2 - \sum_{i=1}^{n}\log(v_i) + (b-1)\sum_{i=1}^{n}\frac{s_i/\log(\frac{\lambda}{y_i}) + (1-u_i)^a\log(v_i)}{v_i^a - (1-u_i)^a}$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log[1 - v_i^{-a}(1-u_i)^a]$$

$$\frac{\partial \log L}{\partial p} = \frac{n}{p} - (a+1)\sum_{i=1}^{n}\frac{u_i}{v_i} + (b-1)\sum_{i=1}^{n}\frac{a u_i v_i^{-(a+1)}(1-u_i)^a}{1 - v_i^{-a}(1-u_i)^a}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{2n}{\lambda} - a\sum_{i=1}^{n} t_i(1-u_i)^{-1} + (a+1)\sum_{i=1}^{n}\frac{(1-p)t_i}{v_i} + a(b-1)\sum_{i=1}^{n}\frac{t_i(1-u_i)^{a-1}}{v_i^a - (1-u_i)^a}\left[1 - (1-p)v_i^{-1}(1-u_i)\right]$$

It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the log-likelihood function. For interval estimation of the model parameters, we require 4 x 4 observed information matrix J(θ) = {Jrs} (for r; s = a, b, p, λ). Under standard regularity conditions, the multivariate normal $N_4(0; J(\theta)^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, J(θ) is the total observed information matrix evaluated at θ. Therefore, approximate 100(1-φ)% confidence intervals for a, b, p and λ can be determined as:

$$a \pm Z_{\frac{\phi}{2}}\sqrt{\hat{\mathbf{J}}_{aa}},\quad b \pm Z_{\frac{\phi}{2}}\sqrt{\hat{\mathbf{J}}_{bb}},\quad p \pm Z_{\frac{\phi}{2}}\sqrt{\hat{\mathbf{J}}_{pp}},\quad \lambda \pm Z_{\frac{\phi}{2}}\sqrt{\hat{\mathbf{J}}_{\lambda\lambda}} \quad \text{and}$$

where $Z_{\frac{\phi}{2}}$ is the upper ϕ^{th} percentile of the standard normal distribution.

VI. APPLICATION TO REAL DATA

In this section, we analyze one data set to demonstrate how the GIR distribution can be a good life time model in comparison with many known distributions. We consider the data set originally reported by Bjerkedal(1960). This data set consists of 72 observations of survival times guinea pigs injected with different doses of tubercle bacilli. The data set has been considered by several authors in the literature, see, Kundu and Howlader (2010) and Cordeiro et al. (2012). The data set follows:

12, 15, 22, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 67, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 263, 297, 341, 341, 376.

The descriptive statistics of the data is presented in Table 1.

Minimum	Median	Mean	Maximum	SD	Skewness	Kurtosis
12.00	70.00	99.82	376	81.118	1.796	5.614

Table 1: Descriptive statistics of pig data

The distribution of the data is positively skewed and leptokurtic. We compare the GIR distribution with the following life time distributions:

1. Inverse exponential (IE) distribution having the pdf

$$g(y;\lambda) = \lambda y^{-2} e^{-(\frac{\lambda}{y})}; \quad \lambda > 0,\ y > 0.$$

2. Inverse Rayleigh (IR) distribution having the pdf

$$g(y;\lambda) = 2\lambda^2 y^{-3} e^{-(\frac{\lambda}{y})^2}; \quad \lambda > 0,\ y > 0.$$

3. Weibull (W) distribution having the pdf

$$g(y;\lambda,\beta) = \lambda^\beta \beta y^{\beta-1} e^{-(\lambda y)^\beta}; \quad \lambda,\beta > 0,\ y > 0.$$

4. Inverse Weibull (IW) distribution having the pdf

$$g(y;\lambda,\beta) = \lambda^\beta \beta y^{-(\beta+1)} e^{-(\frac{\lambda}{y})^\beta}; \quad \lambda,\beta > 0,\ y > 0.$$

5. Exponentiated inverse Rayleigh (EIR) distribution having the pdf

$$g(y;a,\lambda) = 2a\frac{\lambda^2}{y^3} e^{-(a+1)(\frac{\lambda}{y})^2}; \quad a > 0\ \lambda > 0,\ y > 0.$$

6. Generalized inverse Rayleigh (GR) distribution having the pdf

$$g(y;b,\lambda) = \frac{2b}{\lambda^2 y^3} e^{-(\frac{1}{\lambda y})^2}\left[1 - e^{-(\frac{1}{\lambda y})^2}\right]^{b-1}; \quad b > 0,\ \lambda > 0, y > 0.$$

7. Kumaraswamy inverse Rayleigh (Kw-IR) distribution having the pdf

$$g(y;a,b,\lambda) = \frac{2ab\lambda^2}{y^3} e^{-a(\frac{\lambda}{y})^2}\left[1 - e^{-a(\frac{\lambda}{y})^2}\right]^{b-1}; \quad a > 0,\ b > 0,\ \lambda > 0.$$

8. Kumaraswamy Marshall-Olkin inverse exponential (Kw-MOIE) distribution having the pdf

$$\begin{aligned} g(y;a,b,\alpha,\lambda) \;=\; & ab\alpha\lambda x^{-2} e^{-a(\frac{\lambda}{y})}[\alpha + (1-\alpha)e^{-(\frac{\lambda}{y})}]^{-a-1} \\ & [1 - e^{-a(\frac{\lambda}{y})}[\alpha + (1-\alpha)e^{-(\frac{\lambda}{y})}]^{-a}]^{b-1}; \quad a,b,\alpha,\lambda > 0,\ y > 0. \end{aligned}$$

9. Kumaraswamy Fréchet (Kw-Fr) distribution having the pdf

$$g(y;a,b,\lambda,\beta) = ab\beta\lambda^\beta y^{-\beta-1} e^{-a(\frac{\lambda}{y})^\beta}[1 - e^{-a(\frac{\lambda}{y})^\beta}]^{b-1}; \quad a,b,\lambda,\beta > 0,\ y > 0.$$

The parameters is estimated numerically using maximum likelihood estimate method. The values of log-likelihood (-log L), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQCI) statistics for GIW and their sub-models are calculated. The better distribution corresponds to smaller -log L, AIC, CAIC, BIC and HQIC.

We apply the Crammer-von Mises(W^*) and Anderson-Darling (A^*) statistic for formal goodness-of-fit to verify which distribution fits better to this data. In general, the smaller the values of the statistics W^* and A^*, shows better the fit to the data.

The values of estimates, -log L, AIC, CAIC, BIC, HQIC are listed in Table 2 and W^*; A^*, K-S and p-values for all models are listed in Table 3.

Model	Parameters	-log L	AIC	CAIC	BIC	HQIC
IE	λ =60.0975	402.6718	807.3436	807.4007	809.6203	808.2499
IR	λ =46.7748	406.7361	815.4722	815.5293	817.7489	816.3785
W	λ =0.0090, β= 1.3932	397.1477	798.2954	798.4693	802.8487	800.1081
IW	λ =54.1888, β= 1.4148	395.6491	795.2982	795.4721	799.8515	797.1109
EIR	α=3936.64, λ = 0.745	407.7350	817.4700	817.6400	822.0300	819.7300
GR	B=0.616, λ = 0.025	400.9100	807.8200	808.0100	810.3800	808.4590
Kw-IR	α=8.18, b=0.616 λ = 13.65	400.9150	807.8200	808.1800	814.6500	813.1370
Kw-MOIE	a=68.1393, b=2.6258 α=8.8727 λ = 0.1758	391.3500	790.7000	791.3000	799.8000	794.3000
Kw-Fr	a=45.7326, b=8.2723 λ = 0.7111, β= 0.6207	390.2500	788.5000	789.1000	797.6000	792.1000
GIR	a=54.368, b=308.470 p=0.068, λ = 69.693	385.4300	778.8612	780.5421	792.9345	789.1472

Table 2: MLE's of the parameters and some measures for the fitted models.

Model	W*	A*	K-S	p-value
IE	0.8322	4.5927	0.1847	0.0148
IR	1.2574	6.4936	0.2361	0.0943
W	0.4325	2.3763	0.1468	0.0989
IW	0.2521	1.5017	1.1520	0.0718
EIR	0.1362	0.7546	0.2370	0.0004
GR	0.1958	1.1447	0.1964	0.3516
Kw-IR	0.1030	0.6260	0.0900	0.3634
Kw-MOIE	0.1219	0.7404	0.1181	0.4167
Kw-Fr	0.1160.	0.7044	0.1004	0.5623
GIR	0.0929	0.6964	0.0986	0.6034

Table 3: Goodness-of-fit statistic for various models fitted to pig data.

From the Table 2 and Table 3, we can see that, GIR distribution has smallest -log L, AIC, CAIC, BIC, HQIC, W_*, A_* and K -S values. Also the GIR distribution has highest p-value. Hence the new model, that is GIR distribution, yields a better fit than the other models for this data set.

The fitted density and the empirical cdf plot of the GIR distribution are presented in Figure 3. The figure indicates a satisfactory fit for the GIR distribution.

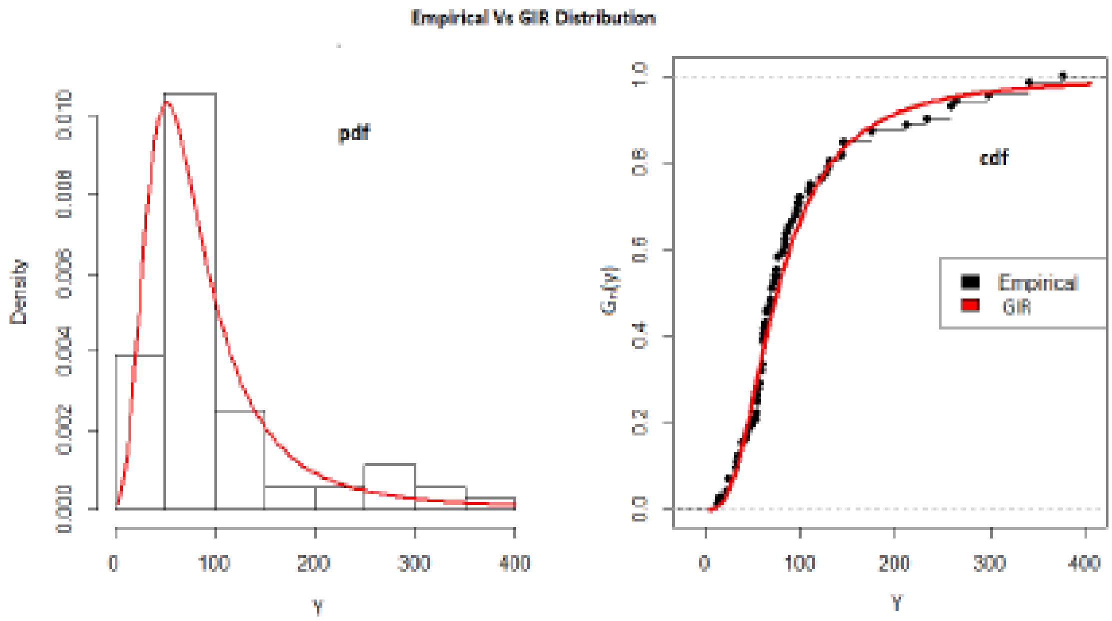

Figure 3: Plots of the estimated pdf and cdf of the GIR model for pig data

To test the null hypothesis H_0: Kw-IR versus H_1: GIR or equivalently H_0: $p = 1$ versus H_1: $p \neq 1$, we use likelihood ratio test statistic whose value is 5.7406 (p-value 0.0254). As a result, the null model Kw-IR is rejected in favour of alternative model GIR at any significant level greater than 0.0254.

VII. CONCLUSION

Using minimax geometric extreme stable concept, introduced a new family of distributions namely Kumaraswamy Inverse discrete Linnik G family of distributions. By suitable the values of the parameters, we will obtain Kumaraswamy inverse truncated discrete Mittag-Leffler G family of distributions, Kumaraswamy inverse truncated negative binomial G distribution, Kumaraswamy inverse Marshall-Olkin G family of distributions etc. In this paper, We consider one member of the family and the base line distribution as Rayleigh distribution. The density function can be given mixture of inverse Rayleigh distribution. The explicit expression for the ordinary moments, quantiles and moments of residual life. Most commonly used best method of estimation, that is maximum likelihood method is adopted to find the parameters. For real life application, we consider one data set which is compatible with nine other models and to convince the adaptability of our proposed model. We have to study the model identifiability, general properties of the proposed distribution such as mean deviation, entropy, Stochastic ordering, order statistics, characterization etc. We believe that the proposed model will widespread applicability in addressing real-world problems across various disciplines including medicine, engineering and the social sciences. Also the researchers extend the study various lifetime distributions as base line distribution.

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