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f - Prime Radical in Ternary Semigroup

Harish Ganesh Nemade

Associate Professor
Department of Mathematics
TVES's Dhanaji Nana Mahavidyalaya, Faizpur, India

Abstract: In this paper we study some results on f-prime ideals f-completely prime ideals, f-prime radical and f-completely prime radical in ternary semigroups. Here we prove the results, (i) An ideal P of ternary semigroup S is completely f-prime if and only if $S\backslash P$ is ternary sub-semigroup or empty. (ii) If S is an idempotent ternary semigroup, then every maximal ideal M of S is a f-prime ideal.

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I. Introduction

The concept of a semigroup is very simple and plays a large role in the development of Mathematics. The theory of semigroups is similar to group theory and ring theory. The algebraic theory of semigroups was developed by CLIFFORD [3] and PRESTON [9]. Here we generalised the structure for ternary semigroup is studied by M.L. Santiago and S. Sri Bala [11] and prime ideals in ternary semigroup was introduced by S. Bashir and M. Shabir [1], here we start with some basic definitions that are required to give the results.

Definition 1.1. A System S with a ternary operation $*$ is called ternary semigroup if $*$ is associative operation.

Definition 1.2. A ternary semigroup S is said to be finite if the cardinality of S is finite.

Definition 1.3. A ternary semigroup S is said to be commutative if $a*(b*c) = (a*b)*c$, for all $a, b, c \in S$.

Definition 1.4. An element e of a commutative ternary semigroup S is said to be identity of S if, $e*e*a = a$, for all $a \in S$.

Definition 1.5. A nonempty subset A of a ternary semigroup S is said to be a ternary sub-semigroup of S if

for $a, b, c \in A \Rightarrow a*b*c \in A$.

Note. For convenience we write ab instead of $a*b$.

Definition 1.6. A nonempty subset $I$of a ternary semigroup S is said to be a left ideal of S if $ISS \subseteq I$.

Definition 1.7. A nonempty subset $I$of a ternary semigroup S is said to be a right ideal of S if $SSI \subseteq I$.

Definition 1.8. A nonempty subset $I$of a ternary semigroup S is said to be a lateral ideal of S if $SIS \subseteq I$.

Definition 1.9. A nonempty subset $I$of a ternary semigroup S is said to be an ideal of S if it is left, right and lateral ideal.

Definition 1.10. A proper ideal M of a ternary semigroup S is said to be a maximal ideal if it is not properly contained in any proper ideal of S.

Definition 1.11. An ideal I of a ternary semigroup S is said to be an idempotent ideal if $I^2 = I$.

Definition 1.13. A ternary semigroup S is said to be an idempotent ternary semigroup if $S^3 = S$.

Definition 1.14. An ideal A of a commutative ternary semigroup S is said to be a principal ideal if A is generated by single element.

Definition 1.15. an element a of a ternary semigroup S is said to be idempotent if $a^3 = a$.

Definition 1.16. An ideal P of a ternary semigroup S is said to be completely prime if

$a, b, c \in S, abc \in P$ implies $a \in P$ or $b \in P$ or $c \in P$.

Definition 1.17. An ideal P of a ternary semigroup S is said to be prime if A, B, C are ideals in S with $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

Definition 1.18. If A is an ideal of a ternary semigroup S, then the intersection of all prime ideals containing A is called prime radical of A and it is denoted by $rad(A)$ or $\sqrt{A}$.

Definition 1.19. If A is an ideal of a ternary semigroup S, then the intersection of all completely prime ideals containing A is called completely prime radical of A and it is denoted by $c.rad(A)$

II. f-prime radical and completely f-prime radicals

Throughout this paper S denotes ternary semigroup and f is a function from S into S such that,

i) $x \in f(S)$ implies $f(x) \subseteq f(S)$;

ii) A is an ideal in S implies $f(x)$ is an ideal in S.

Definition 2.1. An ideal P of S is said to be f-prime ideal if A, B, C are ideals in S with
$f(A).f(B).f(C) \subseteq P$ implies $f(A) \subseteq P$ or $f(B) \subseteq P$ or $f(C) \subseteq P$, were
$f(A) = \cup_{x\in A} f(x)$, $f(B) = \cup_{x\in B} f(x)$, $f(C) = \cup_{x\in C} f(x)$.

Definition 2.2. An ideal P of S is said to be completely f-prime if $f(x), f(y), f(z) \subseteq S$ with
$f(x).f(y).f(z) \subseteq P$ implies $x \in P$ or $y \in P$ or $z \in P$.

Theorem 2.3. An ideal P of S is completely f-prime if and only if $S\backslash P$ is ternary sub-semigroup or empty.

Proof. Let P be a completely f-prime ideal of S with $S\backslash P \neq \phi$.

Let $a, b, c \in S$ with $a \notin P, b \notin P, c \notin P$.

Suppose that, $f(a), f(b), f(c) \subseteq S\backslash P$, then $f(a) \nsubseteq P, f(b) \nsubseteq P, f(c) \nsubseteq P$.

If possible, suppose that $f(a).f(b).f(c) \subseteq P$.

As P is completely f-prime either $a \in P$ or $b \in P$ or $c \in P$. A contradiction.

Hence $f(a).f(b).f(c) \nsubseteq P$ i.e. $f(a).f(b).f(c) \subseteq S\backslash P$.

Thus $S\backslash P$ is ternary sub-semigroup.

Conversely, suppose that $S\backslash P$ is ternary sub-semigroup of S or empty.

If $S\backslash P$ is empty, then $P = S$, hence P is completely f-prime.

Suppose that $S\backslash P$ is ternary sub-semigroup of S.

Let $f(a).f(b).f(c) \subseteq P$, where $f(a), f(b), f(c) \subseteq S$.

Suppose that $f(a), f(b), f(c) \subseteq P$, then $f(a), f(b), f(c) \subseteq S\backslash P$.

As $S\backslash P$ is ternary sub-semigroup, $f(a).f(b).f(c) \subseteq S\backslash P$. A contradiction.

Hence either $f(a) \subseteq P$ of $f(b) \subseteq P$ or $f(c) \subseteq P$.

Therefore $a \in P$ or $b \in P$ or $c \in P$. Hence P is completely f-prime ideal of S.

Theorem 2.4. Every completely f-prime ideal in S is f-prime.

Proof. Let P be completely f-prime ideal in ternary semigroup S.

Let A, B, C be ideals in S such that $f(A).f(B).f(C) \subseteq P$.

If possible, suppose that, $f(a) \nsubseteq P, f(b) \nsubseteq P$ and $f(c) \nsubseteq P$.

Then there exists $x \in f(a)\backslash P, y \in f(b)\backslash P$ and $z \in f(c)\backslash P$

Now $f(a).f(b).f(c) \subseteq f(A)f(B)f(C) \subseteq P$.

As P is completely f-prime, $f(a) \subseteq P$ or $f(b) \subseteq P$ or $f(c) \subseteq P$.

Therefore, $x \in f(a) \subseteq P$ or $y \in f(b) \subseteq P$ or $z \in f(c) \subseteq P$. A contradiction.

Therefore $f(a) \subseteq P, \forall a \in A$ or $f(b) \subseteq P, \forall b \in B$ or $f(c) \subseteq P, \forall c \in C$.

So $f(A) \subseteq P$ or $f(B) \subseteq P$ or $f(C) \subseteq P$.

Hence P is f-prime ideal/

Theorem 2.5. If P is a prime ideal in commutative ternary semigroup S, then the following conditions are equivalent:

1) P is f-prime ideal;

2) If $f(a), f(b), f(c) \subseteq S$ such that $f(a).f(b).f(c) \subseteq P$, then either $f(a) \subseteq P$ or $f(b) \subseteq P$ or $f(c) \subseteq P$.
3) If $f(a), f(b)$ and $f(c)$ are principal ideals in S such that $\big(f(a)\big).\big(f(b)\big).\big(f(c)\big) \subseteq P$, then either $f(a) \subseteq P$ or $f(b) \subseteq P$ or $f(c) \subseteq P$.

Proof. 1) ⇒ 2) Let $f(a), f(b), f(c) \subseteq S$ such that $f(a).S.f(b).S.f(c) \subseteq P$.

So, $S.f(a).S.f(b).S.f(c).S \subseteq SPS \subseteq P$

$$\Rightarrow (S.f(a).S)(S.f(b).S)(S.f(c).S) \subseteq P.$$

Since P is f-prime, either $S.f(a).S \subseteq P$ or $S.f(b).S \subseteq P$ or $S.f(c).S \subseteq P$.

Suppose that, $S.f(a).S \subseteq P$ and $f(A) = \big(f(a)\big)$, then

$\big(f(A)\big)^3 = \big(f(A)\big).\big(f(A)\big).\big(f(A)\big) \subseteq S.f(a).S \subseteq P$.

Therefore, $f(A) \subseteq P \Rightarrow f(A) = f(a) \subseteq P$.

Similarly, if $S.f(b).S \subseteq P \Rightarrow f(b) \subseteq P$ and $S.f(c).S \subseteq P \Rightarrow f(c) \subseteq P$.

Thus either $f(a) \subseteq P$ or $f(b) \subseteq P$ or $f(c) \subseteq P$.

2) ⇒ 3) Let $\big(f(a)\big).\big(f(b)\big).\big(f(c)\big) \subseteq P$

$\Rightarrow f(a).S.f(b).S.f(c) \subseteq \big(f(a)\big).\big(f(b)\big).\big(f(c)\big) \subseteq P$

$\Rightarrow f(a).S.f(b).S.f(c) \subseteq P$

$\Rightarrow f(a) \subseteq P$ or $f(b) \subseteq P$ or $f(c) \subseteq P$.

3) ⇒ 1) Let $f(a), f(b)$ and $f(c)$ are principal ideals of S such that,

$\big(f(a)\big).\big(f(b)\big).\big(f(c)\big) \subseteq P$

$\Rightarrow f(a) \subseteq P$ or $f(b) \subseteq P$ or $f(c) \subseteq P$.

$\Rightarrow P$ is $f$prime ideal of S.

Theorem 2.6. If S is an idempotent ternary semigroup, then every maximal ideal M of S is a f-prime ideal.

Proof. Let M be maximal ideal of S.

Lat A, B and C be ideals of S such that $f(A).f(B).f(C) \subseteq M$.

If possible, suppose that $f(A) \nsubseteq M, f(B) \nsubseteq M$ and $f(C) \nsubseteq M$.

Now, $f(A) \nsubseteq M \Rightarrow M \cup f(A) = S$, similarly $M \cup f(B) = S$ and $M \cup f(C) = S$.

Now as S is idempotent ternary semigroup,

$$S = S^3 = \big(M \cup f(A)\big).\big(M \cup f(B)\big).\big(M \cup f(C)\big)$$
$$= M^3 \cup M^2.f(C) \cup M.f(B).M \cup f(A).M^2 \cup M.f(B).f(C)$$
$$\cup f(A).M.f(C) \cup f(A).f(B).M \cup f(A).f(B).f(C) \subseteq M$$

Hence, $M = S$. A contradiction.

Therefore, $f(A) \subseteq M$ or $f(B) \subseteq M$ or $f(C) \subseteq M$.

Hence M is f-prime ideal.

Definition 2.7. If A is an ideal of S, then the intersection of all completely f-prime ideals containing A is called completely f-prime radical of A and it is denoted by $c.f.rad(A)$.

Definition 2.8. If A is an ideal of S, then the intersection of all f-prime ideals containing A is called f-prime radical of A and it is denoted by $f.rad(A)$.

Theorem 2.9. If A is an ideal of S, then $f.rad\big(f.rad(A)\big) = f.rad(A)$ and

$c.f.rad\big(c.f.rad(A)\big) = c.f.rad(A)$.

Proof. Let A is an ideal of S.

We have, $f.rad(A) =$ The intersection of all f-prime ideals of S containing $A = A^*$ (say).

Now, $f.rad\big(f.rad(A)\big) = f.rad(A^*)$

$=$ The intersection of all f-prime ideals of S containing A^*

$= \text{The intersection of all } f\text{-prime ideals of } S \text{ containing } A$

$= A^* = f.rad(A)$.

Similarly, $c.f.rad\big(c.f.rad(A)\big) = c.f.rad(A)$.

Theorem 2.10. Let A and B be two ideals of S. If $A \subseteq B$, then $f.rad(A) \subseteq f.rad(B)$ and $c.f.rad(A) \subseteq c.f.rad(B)$.

Proof. Let $A \subseteq B$, and P be f-prime ideal containing B, then P is f-prime ideal containing A.

Therefore $f.rad(A) \subseteq f.rad(B)$.

Similarly, $c.f.rad(A) \subseteq c.f.rad(B)$.

Theorem 2.11. If A, B and C are ideals of S, then

$f.rad(A.B.C) = f.rad(A \cap B \cap C) = f.rad(A) \cap f.rad(B) \cap f.rad(C)$ and

$c.f.rad(A.B.C) = c.f.rad(A \cap B \cap C) = c.f.rad(A) \cap c.f.rad(B) \cap c.f.rad(C)$.

Proof. Let P be a f-prime ideal of S containing $A.B.C$.

As P is f-prime $\Rightarrow f(A).f(B).f(C) \subseteq P$

$\Rightarrow f(A) \subseteq P$ or $f(B) \subseteq P$ or $f(C) \subseteq P$.

$\Rightarrow A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

$\Rightarrow A \cap B \cap C \subseteq P$.

Therefore P is f-prime ideal containing $A \cap B \cap C$.

Therefore $f.rad(A \cap B \cap C) \subseteq f.rad(A.B.C) - - - - - - - - (I)$

Now, let P is f-prime ideal containing $A \cap B \cap C$.

Then $A.B.C \subseteq P \Rightarrow A.B.C \subseteq A \cap B \cap C \subseteq P$.

Hence P is f-prime ideal containing $A.B.C$.

Therefore, $f.rad(A.B.C) \subseteq f.rad(A \cap B \cap C) - - - - - - - - (II)$

Thus from (I) and (II), $f.rad(A.B.C) = f.rad(A \cap B \cap C) - - - - - (III)$

Now $A.B.C \subseteq A, A.B.C \subseteq B$ and $A.B.C \subseteq C$

$\Rightarrow f.rad(A.B.C) \subseteq f.rad(A), f.rad(A.B.C) \subseteq f.rad(B)$ and $f.rad(A.B.C) \subseteq f.rad(C)$

$\Rightarrow f.rad(A.B.C) \subseteq f.rad(A) \cap f.rad(B) \cap f.rad(C) - - - - - - (IV)$

Let $x \in f.rad(A) \cap f.rad(B) \cap f.rad(C)$.

Therefore, $x \in f.rad(A),\ x \in f.rad(B)$ and $x \in f.rad(C)$.

If possible, suppose that $x \notin f.rad(A.B.C)$.

Then there exists a f-prime ideal P containing $A.B.C$ with $x \notin P$.

Now $A.B.C \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

If $A \subseteq P$, as $x \notin P \Rightarrow x \notin A \Rightarrow x \notin f.rad(A)$.

Similarly, $x \notin f.rad(B)$ and $x \notin f.rad(C)$. A contradiction.

Therefore, $x \in f.rad(A.B.C)$

Therefore, $f.rad(A) \cap f.rad(B) \cap f.rad(C) \subseteq f.rad(A.B.C). - - - - - - - (V)$

So, by (IV) and (V)

$f.rad(A.B.C) = f.rad(A) \cap f.rad(B) \cap f.rad(C)$.

Hence, we get,

$f.rad(A.B.C) = f.rad(A \cap B \cap C) = f.rad(A) \cap f.rad(B) \cap f.rad(C)$

Similarly, $c.f.rad(A.B.C) = c.f.rad(A \cap B \cap C) = c.f.rad(A) \cap c.f.rad(B) \cap c.f.rad(C)$.

Theorem 2.12. If P is a f-prime ideal of S, then $f.rad(P^{2n-1}) = P$, were n is a positive integer.

Proof. We prove this result by applying mathematical induction on n.

Since P is f-prime ideal $P \subseteq f.rad(P) \subseteq P \Rightarrow f.rad(P) = P$.

Hence result holds for $n = 1$.

Suppose that, $f.rad(P^{2k-1}) = P$ for all positive integers $k \leq n$.

Now, $f.rad(P^{2k+1}) = f.rad(P^k.P^k.P)$

$= f.rad(P^k) \cap f.rad(P^k) \cap f.rad(P)$

$= f.rad(P) \cap f.rad(P) \cap f.rad(P) = f.rad(P).$

Hence, result holds for $n = 2k+1$.

Thus, by mathematical induction, $f.rad(P^{2n-1}) = P$, were n is a positive integer.

Theorem 2.13. In a ternary semigroup with identity there is a unique maximal ideal M such that, $f.rad(M^{2n-1}) = M,$ for all $n \in \mathbb{N}$.

Proof. Since $e \in S, e.e.S = S \Rightarrow S^3 = S \Rightarrow S$ is idempotent ternary semigroup.

Since M is maximal ideal in $S,$ by theorem 2.6, M is f-prime ideal. Hence by Theorem 2.12, $f.rad(M^{2n-1}) = M,$ for all $n \in \mathbb{N}$.

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