JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

Relation of Cardinality between Any Set and its Power Set

Vivek Sinha [1], Jaydeep Parejiya [2]
[1] Research Scholar and Lecturer, GTU and Government Polytechnic Diu, India viveksinha7885@gmail.com
2 Government Polytechnic Rajkot, India parejiyajay@gmail.com

Abstract:
This paper proposes an algorithm to derive a general formula for counting the total number of onto functions from a set A with cardinality m to a set B with cardinality m, and verification of Cantor theorem. Let f: A→B is a function such that $|A| = m$ and $|B| = n$, where A and B are finite and non-empty sets, m and n are finite integer values. To count the total number of onto functions using “Stirling number of the second kind”, with the help of Inclusion-exclusion principal. This paper will help for counting all possible onto mappings by means of partition method and will provide the direct count on onto functions using the formula derived in it.
Keywords: Set, function, onto –function, Binomial expansion, cardinality

Introduction:

One of the results in the Set theory is that Cantor’s theorem which says “If A is any set, then ∄ any surjection (onto) of A onto set $P(A)$ of all subsets of A." In fact Cantor proved a more general theorem: for any set X, the cardinality of X is strictly less than the cardinality of the power set of X” [1]

Cantor is known as the founder of modern set theory and he was the first to study the concept of infinite set in attentive detail. In 1874 he proved that Q is countable set and in disparity, that R is uncountable, and arising a question that there are two types of infinity. Cantor’s theorem on sets of subsets shown there are many different orders of infinity and this led him to create a theory of “transfinite” numbers that he published in 1895 and 1897 [2].

One of the greatest revolutions in mathematics occurred when Georg Cantor (1845-1918) promulgated his theory of transfinite sets. Set theory has been widely adopted in mathematics and philosophy.

Least of upper bound is supremum or least upper bound. Greatest of lower bounds is known as greatest lower bound or infimum. A number "p" is limit point, if every neighborhood of “p” contains infinite number of points of the sequence. Limit Point of the sequence need not be a member of the sequence. A set "A" of real number is dence provided between any two real number their lies a member of A. (Irrational/Rational number are dense in R. Every set “E” of real number, bounded below has an infimum. Each real number is supremum of a set of rational number and also for irrational number.

Literature Review

Some important types of function like relation are a subset of Cartesian product of set A and B defined as $R \subseteq A \times B$. In case of Function, let A and B be two sets. then a function from A to B is a set denoted by f of the ordered pair $A \times B$.

1. One-One function- f is known as one-one function if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
2. Onto function- f is known as onto function (map $A\ onto\ B$), if range $R(f) = B$.
3. Any set is called bijection function, if the function is One-One and onto function.
4. Denumerable set-If ∃ a bijection between set A and set of N, then A is known denumerable set.
5. Power set – The collection of all subset of any set "A" is known as power set of A, and is denoted by $P(A)$

We are using finite set, empty set, Infinite set.

Onto function has used mostly for mapping different object of any set with another set like different men can enter into different houses. with the condition that number of houses is equal or less with the number of men. So how can we fit the onto function or, how many ways can four (or more) different men enter into three indistinguishable houses, when each house can contain any number of men?

Solution: We will solve this problem by enumerating all the ways these men's can be placed into the houses. We represent the four men's by M_1, M_2, M_3, and M_4.

(1) First, we note that we can distribute 4 men's so that all four are put into one house,
(2) three are put into one house and a fourth is put into a second house,
(3) two men are put into one house and two put into a second house, and finally,
(4) two are put into one house, and one each put into the other two house.

Each way to distribute these men to these houses can be represented by a way to partition the elements M_1, M_2, M_3, and M_4 into disjoint subsets.

We can put all four men into one house in exactly one way, represented by $M_1M_2M_3M_4$. We can put three men into one house and the fourth men into a different house in exactly four ways, represented by $M_1M_2M_3, M_4$, $M_1M_2M_4, M_3$, $M_1M_3M_4, M_2$ and $M_4M_2M_3$, M_1. We can put two men into one house and two into a second house in exactly three ways, represented by M_1M_2, M_3M_4, M_1M_3, M_2M_4, and M_1M_4, M_2M_3.

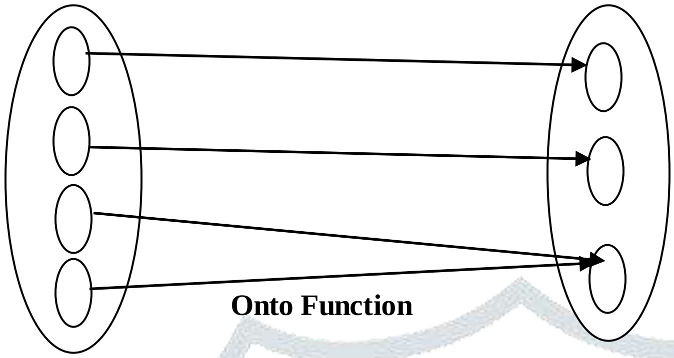
Finally, we can put two men into one house, and one each into each of the remaining two houses in six ways, represented by M_1M_2, M_3,M_4, M_1M_3, M_2,M_4, M_1M_4, M_2, M_3, M_2M_3, M_1, M_4, M_2M_4, M_1, $M_3$and M_3M_4, M_1, M_2[2].

The Stirling formula of the second kind has discussed by $\sum_{r=1}^{n} nc_{r-1}(-1)^{r-1}\{(n-\overline{r-1})^m$ [3].

For finite set, while for infinite set, continuum hypothesis has used as, if set A is not a denumerable set, then $|A| = c$ and $|P(A)| = 2^c$

By continuum hypothesis, $c < 2^c$. i.e. $|A| < |P(A)|$ [4], and the important formula has discussed in [5] as the generating function as $(e^x - 1)^r = r!\sum_{n=r}^{\infty} S(n,r)\frac{x^n}{r!}$.also Stirling numbers of the second kind show up more often than those of the other variety, so let's consider The symbol $\{i\}$ stands for the number of ways to partition a set of n things into k nonempty subsets. For example, there are seven ways to split a four-element set into two parts: {1,2,3}∪{4}, {1,2,4}∪{3}, thus {1,3,4}∪{2}, {2,3,4}∪ {1}, {1,2}∪ {3,4}, {1,3}∪ {2,4}, {1,4}∪ {2,3}[6].

The principle of inclusion–exclusion, especially when it is used to count the number of elements in the union of two sets. Suppose that X and Y are sets. Then, there are X ways to select an element from X and Y ways to select an element from Y. The number of ways to select an element from X or from Y, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from X and the number of ways to select an element from Y, minus the number of ways to select an element that is in both X and Y. Because there are X ∪ Y ways to select an element in either X or in Y and $X \cap Y$ ways to select an element common to both sets, we have $|X \cup Y| = |X| + |Y| - |X \cap Y|$ [7] and for three sets X ,Y and Z $|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |Y \cap Z| - |X \cap Z| + |X \cap Y \cap Z|$ [8]. Cantor could not find any sets whose cardinalities were greater than N_0 but less than $2N_0$, so Cantor hypothesized that $22N_0$ is actually the next cardinal after N_0, i.e. $22N_0$= $2N_1$[9]. Some more work has done in Stirling number as well So, obviously there arise a question is there any possibility to make onto function or if set A has less or more element as compare to set B? That's why Cantor's theorem is important here for getting these types of relation Results - Cantor's Theorem implies that there is an unending progression of larger and larger sets [10]. In particular, it implies that the collection $P(N)$ of all subsets of the natural numbers N is uncountable.



Onto Function

Results - Cantor's Theorem implies that there is an unending progression of larger and larger sets. In particular, it implies that the collection $P(N)$ of all subsets of the natural numbers N is uncountable.

Cantor's theorem: - "If A is any set, then $\nexists$ any surjection (onto) of A onto set $P(A)$ of all subsets of A."

Proposed Method:

Observation:

1. We can check when $m < n$, then no onto function can be mapped.
2. We have element of $P(A)$ always greater than to Set A.
3. If $|A| = m$ and $|B| = n$, total number of functions counted as n^m.
4. Number of one-one function can be assigned by the rule as
   First element of Set A has n options for mapping of set B
   2nd element of Set A has $n - 1$ options for mapping of set B
   3rd element of Set A has $n - 2$ options for mapping of set B
   And similarly, for the m^{th} element of Set A has $\{n - (\overline{m - 1})\}$options for mapping of set B
   So, total number of one -one function will be $n(n-1)(n-2)(n-3(n-4).......(n-\overline{m-1}) = \frac{n!}{(n-m)!}$

   Let A and B be two finite sets, respectively, of m and n distinct elements with $m > n$; and consider all mappings from the set A to the set B. Here We will discuss about the 'Stirling Number of the Second Kind' and determines the numbers of all 'onto 'type and all 'into 'type functions from set A to set B. We will derive the expression for $S\ (m, n)$ using the properties of these functions.
   First, note that the number of all possible distinct functions from A to B is clearly n^m.
   Suppose now that we partition set A into n nonempty disjoint blocks (or unordered subsets) and then connect these blocks (or their individual elements) one-to-one to each of the n elements of set B. One such partition, upon permuting the n distinct elements of set B, leads to $n!$ 'onto' mappings. Since $S\ (m, n)$ is the number of such partitions of A, the total number of all distinct' onto 'mappings from A to B turns out to be $n!\ S\ (m, n)$.
   Accordingly, while the total number of all possible distinct functions from A to B is n^m, the number of all 'onto' type function from A to B is only $n!\ S\ (m, n)$ [2]
   Further, note that the total number of 'into' type function from A to B, (those whose range misses at least one element of B), can be obtained from the inclusion-Exclusion Theorem as $nC_1(n-1)^m - nC_2(n-2)^m + nC_3(n-3)^m - ... + (-1)^n nC_{n-1}(1)^m$.

5. Number of onto function= $\sum_{r=1}^{n} nc_{r-1}(-1)^{r-1}\{(n-\overline{r-1})^m$
6. No. of onto function $= n^m - \frac{n!}{(n-1)!1!}(n-1)^m + \frac{n!}{(n-2)!2!}(n-2)^m - \frac{n!}{(n-3)!3!}(n-3)^m.....$

Case-I: Now taking $|A| = m = 2$, then $|P(A)| = n = 2^2 = 4$

Using observation no. 3

No. of onto function = $4^2 - \frac{4!}{(4-1)!1!}(4-1)^2 + \frac{4!}{(4-2)!2!}(4-2)^2 - \frac{4!}{(4-3)!3!}(4-3)^2 + \frac{4!}{(4-4)!4!}(4-4)^2$

$= 16 - 4(3)^2 + 6(4-2)^2 - 4(4-3)^2 + 0$
$= 16 - 4 \times 9 + 6 \times 4 - 4 \times 1$
$= 16 - 36 + 24 - 4$
$= 0$

Which shows ∄ any surjection from set A onto P(A).

Case-II: when set A is an empty set, i.e. $A = \{ \ \}$, then $|A| = m = 0$

Then power set of A will be $P(A) = \{\{ \ \}\}$ Hence, $|P(A)| = n = 2^0 = 1$

Number of one-one function= $\frac{n!}{(n-m)!} = \frac{1!}{(1-0)!} = \frac{1!}{(1)!} = 1$

While No. of onto function = $1^0 - \frac{1!}{(1-1)!0!}(1-1)^0 + \frac{1!}{(1-2)!2!}(1-2)^0 - \frac{1!}{(1-3)!3!}(1-3)^0$-…….

$= 1 - 1(0^0) \ (0^0 = 1)$
$= 1 - 1$
$= 0$

(We have the binomial expansion as:

$$(1-x)^k = 1 - kx + \frac{k(k-1)}{2!}x^2 - \frac{k(k-1)(k-2)}{3!}x^2 + \cdots$$

Taking $k = 0$,

$$(1-x)^0 = 1 - 0x + \frac{0(0-1)}{2!}x^2 - \frac{0(0-1)(0-2)}{3!}x^2 + \cdots$$

$\Rightarrow \ (1-x)^0 = 1$

Also taking $x = 1$, $\Rightarrow (1-1)^0 = 1$

Hence, we get $0^0 = 1$.)

Which shows ∄ any surjection from set A onto P (A).

$0 < 1$, i.e. cardinality of A always less than that of $P(A)$.

i.e. $|A| < |P(A)|$.

Case III: For infinite set A and P (A), suppose A= N (set of natural number)

We know $|N| = a$ and $|P(N)| = 2^a = c$

and by continuum hypothesis, $a < c$.

And if set A is not a denumerable set, then $|A| = c$ and $|P(A)| = 2^c$

By continuum hypothesis, $c < 2^c$. i.e. $|A| < |P(A)|$ (The continuum hypothesis, introduced by mathematician George Cantor in 1877).[4]

Conclusion:

This paper has proposed an algorithm for the number of onto functions from a set A to the power set (A) . All in three cases we have proved that ∄ any onto function from the set A to the power set $P(A)$ .Which shows cardinality of any set is always less than the cardinality of its power set. The proposed algorithm is valid for all finite, empty and infinite set A.

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