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On the Results of the Extended Fractional Mellin Transform

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Abstract : The Mellin Transform is a powerful tool in mathematics. The Mellin Transform is closely related to other integral transforms, such as the Laplace and Fourier transforms. The Mellin Transform is characterized by several key properties that make it invaluable in many fields. Mellin transforms are a powerful mathematical tool used in a variety of engineering and applied sciences fields. They are particularly useful in solving complex differential equations, analysing asymptotic behaviours and in the field of signal processing.
We introduce a generalized Mellin transformation and an extended fractional Mellin transform in a general form that encompasses the generalized Mellin transformations found in the literature. In this article we have discussed some results which may be used to solve wave equations, Schrodinger's equations , differential equations and etc.

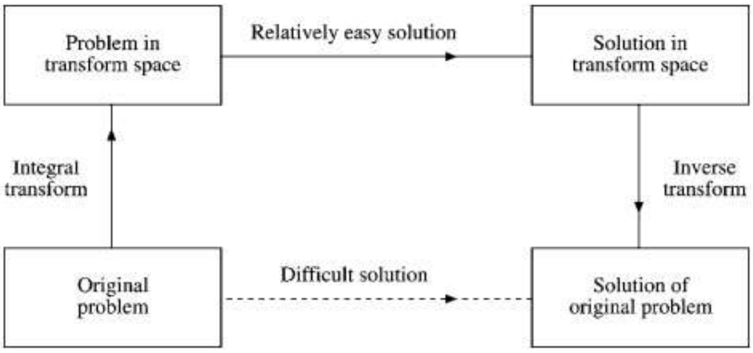
***IndexTerms* - Mellin transform, Fractional Mellin transform, Extended Fractional Mellin transform.**

I. INTRODUCTION

1.1 INTEGRAL TRANSFORMS: Frequently in mathematical physics we encounter pairs of functions related by an expression of the form

$$g(\alpha) = \int_a^b f(t)K(t,\alpha)dt$$

The function $g(\alpha)$ is called the (integral) transform of f (t) by the kernel K(α,t). The operation may also be described as mapping a function f (t) in t-space into another function, g(α), in α-space.
Schematic diagram of Integral transform:



1.2 Mellin transform:

The fractional Mellin Transform generalises Mellin Transform to fractional order . It is defined as

$$M_\alpha[f(x)](s) = \int_0^\infty f(x)x^{s-1}dx$$

1.3 Applications of Mellin transform:

The **Mellin transform**, particularly noted for its **scale invariance**, has found widespread applications across various scientific and engineering disciplines. It is especially useful in **algorithm analysis**, where its ability to handle scaling properties proves advantageous. Beyond this, the Mellin transform is applied in radar systems, stress analysis of wedges, digital audio effects, and signal processing. It also plays a significant role in fractional calculus, special functions, statistical mechanics, cryptography, combinatorics, and the solution of differential equations.

Due to its mathematical versatility, the Mellin transform is a valuable tool in addressing problems in mathematics, physics, and engineering [3]. In the context of 5G network advancements, which promise ultra-reliable, low-latency, and high-capacity communication, novel multimedia applications are emerging. One such application is video steganography, where the Mellin transform has been employed to enhance resistance against deep learning-based steganalysis, thereby improving the security of embedded sensitive information in multimedia content for next-generation networks [4].

I. Mathematical Prerequisite

1.1 The testing function space $E(R^n)$

An infinitely differentiable complex valued function $\emptyset$ on R^n belongs to $\mathrm{E}(\mathrm{R}^n)$ if for each compact set $\subset S_a$, where
$S_a = \{u: u \in \mathrm{R}^n, |u| \le c,\ c > 0\}$,
$\gamma_{E_l}(\emptyset) = \sup_{u \in J} |D_u^l \emptyset(u)| < \infty$.
Thus, $\mathrm{E}(\mathrm{R}^n)$ will denote the space of all $\emptyset \in \mathrm{E}(\mathrm{R}^n)$ with support contained in S_a. Moreover, we say that f is a fractional Mellin transformable, if it is member of E^*, the dual space of E.

1.2 Definition of generalized fractional Fourier transform (FRFT)

The distributional fractional Fourier transform of $f(x) \in E^*(R^n)$, $0 < \alpha \le \frac{\pi}{2}$ is defined by,
$FRFT\{f(x)\} = F_\alpha(p) = \langle f(x), K_\alpha(x,p) \rangle$,
where $K_\alpha(x,p) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2(xp)]}$
where right hand side is meaningful i.e. , $K_\alpha(x,p) \in E$ and $f \in E^*$.

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where right hand side is meaningful i.e. , $K_\alpha(x,p) \in E$ and $f \in E^*$.

1.3 Definition of generalized fractional Mellin transform (FRMT)

The distributional fractional Mellin transform of $f(u) \in E^*(R^n)$, $0 < \theta \le \frac{\pi}{2}$ is defined by,
$FRMT\{f(u)\} = F_\theta(r) = \langle f(u), K_\theta(u,r) \rangle$,
where $K_\theta(u,r) = u^{\frac{2\pi ir}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}(r^2+log^2u)}$,
where right hand side is meaningful i.e. , $K_\theta(u,r) \in E$ and $f \in E^*$.

1.4 An extended fractional Mellin transform

An extended fractional Mellin transform of order α [3] is defined as

$$M_\alpha(r) = EFRMT[f(u)] = \sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty f(u) K_\theta(u,r) du$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty f(u) u^{-ircosec\theta - 1} du , \qquad \text{where } K_\theta(u,r) = u^{-ircosec\theta-1}$$

$$M_\alpha(r) = FRMT[f(u)] = A\int_0^\infty f(u) u^{-ircosec\theta-1} du \qquad \text{where } A = \sqrt{\frac{1-icot\theta}{2\pi}}$$

In our previous work, we have explored the properties of an extended fractional Mellin transform, including linearity, scaling, modulation, differentiation, and Parseval's identity. Additionally, we have derived an inversion formula for the extended fractional Mellin transform.

III. On the Results of the Extended Fractional Mellin Transform

Result: 3.1 If $EFRMT[f(u)] = \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty f(u) K_\theta(u,r) du$ then

Prove that $EFRMT[1](r) = i\,sin\,\theta \sqrt{\frac{1-icot\theta}{2\pi}}$ and $EFRMT[1](r) = 0,\ \theta = 0, \frac{\pi}{2}$.

Proof:
Consider,
LHS= $EFRMT[1](r)$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty (1) K_\theta(u,r) du$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty (1) u^{-ircosec\theta-1} du$$

Putting $\log u = m$

$\therefore u = e^m$

$\therefore du = e^m dm$

If $u = 0 \Rightarrow m = \infty$

If $u = \infty \Rightarrow m = 0$

$$EFRMT[1](r) = \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty (e^m)^{-ircosec\,\theta-1} e^m dm$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty (e)^{-imrcosec\,\theta-1} e^{-m} e^m dm$$

$$= -\sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty (e)^{-imrcosec\,\theta-1} dm$$

$$= -\sqrt{\frac{1-icot\theta}{2\pi}} \left[\frac{e^{-ir\,cosec\theta\, m}}{-ircosec\,\theta}\right]_0^\infty$$

$$= -\sqrt{\frac{1-icot\theta}{2\pi}} \frac{1}{-ircosec\,\theta} [e^{-\infty} - e^0]$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \frac{\sin\theta}{ir} [0-1]$$

$$= i\sin\theta \sqrt{\frac{1-icot\theta}{2\pi}}$$

when $\theta = 0 \Rightarrow EFRMT[1](r) = 0,$

$\theta = \frac{\pi}{2} \Rightarrow EFRMT[1](r) = 0$

Hence proved.

Result: 3.2 If $EFRMT[f(u)] = \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty f(u) K_\theta(u,r) du$ then

Prove that $EFRMT[u^{ia}](r) = \sqrt{\frac{1-icot\theta}{2\pi}} \frac{i}{(r\,cosec\theta - a)}$

Proof:

Consider,

LHS= $EFRMT[u^{ia}](r)$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty (u^{ia}) K_\theta(u,r) du$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty (u^{ia}) u^{-ircosec\theta-1} du$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty u^{-ircosec\theta+ia-1} du$$

Putting $\log u = m$

$\therefore u = e^m$

$\therefore du = e^m dm$

If $u = 0 \Rightarrow m = \infty$

If $u = \infty \Rightarrow m = 0$

$$EFRMT[u^{ia}](r) = \sqrt{\frac{1-icot\alpha}{2\pi}} \int_\infty^0 (e^m)^{-ircosec\theta+ia-1} e^m dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty (e)^{-irmcosec\theta+iam} e^{-m} e^m dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty (e)^{(-ircosec\theta+ia)m}\, dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty (e)^{i(-rcosec\theta+a)m}\, dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{(e)^{-i(rcosec\theta-a)m}}{i(-rcosec\theta+a)}\right]_0^\infty$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{(e)^{-\infty}-(e)^0}{-i(rcosec\theta-a)}\right]$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{0-1}{i(rcosec\theta-a)}\right]$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \frac{i}{(rcosec\theta-a)}$$

Hence proved

Result: 3.3 If $EFRMT[f(u)] = \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty f(u)K_\theta(u,r)du$ then

Prove that $EFRMT\left[e^{i(alogu)}\right](r) = \sqrt{\frac{1-icot\theta}{2\pi}} \frac{i}{(r\,cosec\theta-a)}$

Proof:
Consider,
LHS= $EFRMT\left[e^{i(alogu)}\right](r)$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty \left(e^{i(alogu)}\right)K_\theta(u,r)du$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty \left(e^{i(alogu)}\right)u^{-ircosec\theta-1}du$$

Putting $\log u = m$

$\therefore u = e^m$

$\therefore du = e^m dm$

If $u = 0 \Rightarrow m = \infty$

If $u = \infty \Rightarrow m = 0$

$$EFRMT\left[e^{i(alogu)}\right](r) = \sqrt{\frac{1-icot\alpha}{2\pi}} \int_\infty^0 \left(e^{i(am)}\right)(e^m)^{-ircosec\theta-1}e^m dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty \left(e^{i(am)}\right)(e^m)^{-ircosec\theta-1}e^m dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty \left(e^{i(am)}\right)(e)^{-irmcosec\theta}e^{-m}e^m dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \int_0^\infty (e)^{i(-rcosec\theta+a)m} dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{(e)^{i(-rcosec\theta+a)m}}{i(-rcosec\theta+a)}\right]_0^\infty dm$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{(e)^{-i(rcosec\theta-a)m}}{-i(rcosec\theta-a)}\right]_0^\infty$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{(e)^{-\infty}-e^0}{i(rcosec\theta-a)}\right]$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{0-1}{i(rcosec\theta-a)}\right]$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{i}{i^2(rcosec\theta-a)}\right]$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \left[\frac{i}{(a-rcosec\theta)}\right]$$ Hence proved

Result: 3.4 If $EFRMT[f(u)] = \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty f(u)K_\theta(u,r)du$ then

Prove that $EFRMT\left[e^{ia(logu)^2}\right](r) = \sqrt{\frac{1-icot\theta}{2a}}\ e^{\frac{i}{4}\left[\pi+\frac{r^2cosec^2\theta}{a}\right]}$

Proof:
Consider,
LHS= $EFRMT\left[e^{ia(logu)^2}\right](r)$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty \left(e^{ia(logu)^2}\right)K_\theta(u,r)du$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty \left(e^{ia(logu)^2}\right)u^{-ircosec\theta-1}du$$

Putting $\log u = m$

$\therefore u = e^m$

$\therefore du = e^m dm$

If $u = 0 \Rightarrow m = \infty$

If $u = \infty \Rightarrow m = 0$

$$EFRMT\left[e^{ia(logu)^2}\right](r) = \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty e^{iam^2}(e^m)^{-ircosec\theta-1}e^m dm$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty e^{iam^2}(e^m)^{-ircosec\theta}e^{-m}e^m dm$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty e^{i[am^2-rcosec\theta m]}dm$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \int_0^\infty e^{i[am^2+bm]} dm \quad \text{where, } b = -rcosec\theta$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \frac{e^{i\pi/4}\sqrt{\pi}}{\sqrt{a}} e^{ib^2/4a}$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \frac{e^{i\pi/4}\sqrt{\pi}}{\sqrt{a}} e^{ib^2/4a}$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \frac{e^{i\pi/4}\sqrt{\pi}}{\sqrt{a}} e^{ir^2cosec^2\theta/4a}$$

$$= \sqrt{\frac{1-icot\theta}{2\pi}} \sqrt{\frac{\pi}{a}} e^{i\left[\frac{\pi}{4}+\frac{r^2cosec^2\theta}{4a}\right]}$$

$$= \sqrt{\frac{1-icot\theta}{2a}} e^{\frac{i}{4}\left[\pi+\frac{r^2cosec^2\theta}{a}\right]} \quad \text{Hence Proved}$$

IV. RESULTS AND DISCUSSION

In this article we have discussed some important results of an Extended Frational Mellin Transform which may used to solve wave equations, Schrodinger's equations , differential equations and etc.

S.N.	Function	Result
1	$EFRMT[1](r)$	$i\, sin\, \theta \sqrt{\frac{1-icot\theta}{2\pi}}$ and $EFRMT[1](r) = 0,\ \theta = 0, \frac{\pi}{2}$.
2	$EFRMT[u^{ia}](r)$	$\sqrt{\frac{1-icot\theta}{2\pi}} \frac{i}{(r\, cosec\theta - a)}$
3	$EFRMT\left[e^{i(alogu)}\right](r)$	$\sqrt{\frac{1-icot\theta}{2\pi}} \frac{i}{(r\, cosec\theta - a)}$
4	$EFRMT\left[e^{ia(logu)^2}\right](r)$	$\sqrt{\frac{1-icot\theta}{2a}} e^{\frac{i}{4}\left[\pi+\frac{r^2cosec^2\theta}{a}\right]}$

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