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Common Fixed-Point Theorem using Generalization of Kannan Contractive on S-Metric Space

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Abstract: In this paper, we prove the existence and uniqueness of common fixed points for pairs of self-maps that satisfy the contractive conditions of Kannan types in a complete S-metric space. By applying appropriate contractive condition, we extend known results in fixed point theory and provide new insights into the behavior of such mappings. In addition, an example is also provided to illustrate the existence and uniqueness of common solution. **Keywords:** S-metric space, Common fixed point, Fixed point, Kannan Contraction.

MSC: 47H10, 54H25

1. Introduction

The Banach fixed-point theorem [2], commonly known as the contraction mapping theorem in mathematics, plays a crucial role in studying metric spaces. The Fixed Point Theorem is a fundamental result in mathematics that establishes the conditions under which a point remains unchanged when a function or operator is applied to it. Numerous areas of mathematical analysis depend significantly on this idea. Specifically, in matrix spaces, fixed point theorems offer valuable insights into the behavior of both linear and nonlinear transformations.

In 1968 R. Kannan [8] proposed his contraction as a generalization of the Banach contraction principle to extend fixed point results where the usual distance-based contractive condition was too restrictive. Kannan's work laid the groundwork for exploring nonlinear integral equations**,** optimization problems**,** and operator equations, where interactions were more complex. These theorems allowed researchers to work with more general types of equations, especially those arising in economics, engineering, and control theory.

The concept of b-metric space was first proposed by Bhaktin [1] and in 1993 Czerwik [3] further explored it. A new space known as partial metric space has been developed by Matthews [10] in 1994 and established the fixed point theorem therein. In similar manner, Gońnicki [6] demonstrated F-expanding type mappings followed up by Goswami *et al.* [7] to establish a new type of in a metric space and proved the fixed point results. Wardowski [18], in 2012 demonstrated fixed point results for F-contraction mapping in a complete metric space.

Shabnam Sedghi *et al.* [11] proved the fixed point theorems using weakly contractive mapping in S-metric spaces. Different applications of some contractive conditions have been constructed on S-metric space such as differential equations, complex valued functions etc. Due to its significance, many researchers [12-17] have broadened its scope by obtaining various extensions of fixed point theory in S-metric space.

The main aim of this paper is to propose and prove some common fixed point theorems for functions by employing generalized forms of Kannan contraction in S-metric spaces. The following definitions, notations, basic lemmas and remarks will be needed in the sequel.

2. Prillimaries

Definition 2.1.[5] “Let X be a non-empty set. Then a mapping $d: X\times X\to[0,\infty)$ is called a metric if for all $u,v,w \in X$,

(M_1) $d(u,v)=0$ if and only if $u=v=0$;

(M_2) $d(u,v)=d(v,u)$;

(M_3) $d(u,w) \leq d(u,v)+d(v,w)$.

Then the pair (X, d) is called a metric space.”

Definition 2.2.[11] “Let X be a non-empty set. An S-metric on X is a mapping $S: X^3\to \mathbb{R}_+$ that satisfies the following condition:

(S_1) $S(u,v,w)=0$ if and only if $u=v=w=0$;

(S_2) $S(u,v,w) \leq S(u,u,a)+S(v,v,a)+S(w,w,a)$,

for all $u,v,w,a \in X$. Then the pair (X, S) is called an S-metric space.”

Example 2.3.[11] “Let $X = \mathbb{R}$. $S(u,v,w)=|u-w|+|v-w|$. Then $S(u,v,w)$ is an S-metric on $\mathbb{R}$, which is known as usual S-metric space on X.”

Definition 2.4.[12] “A sequence $\{u_n\}$ in (X, S) is said to be convergent to x, denoted by $\lim_{n\to\infty} u_n = u$ if $u_n\to u$ or $S(u_n, u_n, u) \to 0$ as $n\to \infty$.”

Definition 2.5.[12] “A sequence $\{u_n\}$ in (X, S) is said to be Cauchy sequence if $S(u_n, u_n, u_m)\to 0$ as $n, m\to \infty$.”

Definition 2.6.[12] “An S-metric space (X, S) is said to be complete if every Cauchy sequence in X is convergent.”

Lemma 2.7.[13] “Every sequence $\{u_n\}$ of elements from S-metric space (X, S), having the property that there exists $\lambda \in [0, 1)$ such that $S(u_n, u_n, u_{n+1}) \leq \lambda\, S(u_{n-1}, u_{n-1}, u_n)$ for every $n \in \mathbb{N}$, is a Cauchy.”

Theorem 2.8.[19] Let (X,d) be a complete metric space, T:X→X be a self mapping. Suppose there exist $\gamma \in [0,\frac{1}{2})$ such that

$$d(Tu,Tv) \leq \gamma\, [d(u,Tu)+d(v,Tv)],$$

For all u, v ∈X with u≠v. Then T has a unique fixed point in X.

Theorem 2.9.[4] Let (X,d) be a complete metric space, T_1,T_2:X→X be a pair of self maps. Suppose there exist $\gamma \in [0,\frac{1}{2})$ such that

$$d(T_1u,T_2v) \leq \gamma\, [d(u,T_1u)+d(v,T_2v)],$$

For all u, v ∈X with u≠v. Then T_1 and T_2 has a unique common fixed point in X.

3. Main Result

Theorem 3.1. Let (X, S) be a complete S-metric space and let T_1,T_2: X→X be a pair of self maps such that

$$S(T_1u,T_1u,T_2v) \leq \gamma[S(u,u,T_1u)+S(v,v,T_2v)]\,, \qquad (3.1)$$

For all u, v ∈ X where $\gamma \in [0,\frac{1}{2})$ along with $2\alpha+\beta<1$. Then T_1 and T_2 has a unique common fixed point in X.

Proof: Let $u_0 \in X$ and we define a sequence $\{u_n\}$ such that $u_{2n+1} = T_1u_{2n}$ and $u_{2n+2} = T_2u_{2n+1}$ for $n = 0,1,2,3\ldots$.

If there is $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then from (3.1)we have

$$S(u_{n_0+1},u_{n_0+1},u_{n_0+2})=S(T_1u_{n_0},T_1u_{n_0},T_2u_{n_0+1})$$
$$\leq\gamma\big[S(u_{n_0},u_{n_0},T_1u_{n_0})+S(u_{n_0+1},u_{n_0+1},T_2u_{n_0+1})\big]$$
$$=\gamma\big[S(u_{n_0},u_{n_0},u_{n_0+1})+S(u_{n_0+1},u_{n_0+1},u_{n_0+2})\big]$$
$$=\gamma\big[S(u_{n_0},u_{n_0},u_{n_0})+S(u_{n_0+1},u_{n_0+1},u_{n_0+2})\big]$$
$$=\gamma\big[S(u_{n_0+1},u_{n_0+1},u_{n_0+2})\big],$$

a contradiction, since $\gamma<\frac{1}{2}<1$.

Thus it is permissible to assume that $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, the following cases are investigated.

Case I: n is even, Here $n = 2i$ for some $i = \{0,1,2,3\ldots\}$. From Equation (3.1) we obtain

$$S(u_{2i},u_{2i},u_{2i+1})=S(T_1u_{2i-1},T_1u_{2i-1},T_2u_{2i})$$
$$\leq\gamma[S(u_{2i-1},u_{2i-1},T_1u_{2i-1})+S(u_{2i},u_{2i},T_2u_{2i})]$$
$$=\gamma[S(u_{2i-1},u_{2i-1},u_{2i})]+\beta[S(u_{2i},u_{2i},u_{2i+1})]$$
$$\leq\frac{\gamma}{1-\gamma}[S(u_{2i-1},u_{2i-1},u_{2i})]\,. \qquad (3.2)$$

Case II: n is odd, Here $n = 2i+1$ for some $i = \{0,1,2,3\ldots\}$. Using simillar arguments given in case I, we have

$$S(u_{2i+1},u_{2i+1},u_{2i+2})\leq\frac{\gamma}{1-\gamma}[S(u_{2i},u_{2i},u_{2i+1})]. \qquad (3.3)$$

Combining equation (3.2) and (3.3) together, we obtained

$$S(u_n,u_n,u_{n+1})\leq\frac{\gamma}{1-\gamma}[S(u_{n-1},u_{n-1},u_n)],$$

Thus,

$$S(u_n,u_n,u_{n+1}) \leq \lambda\, [S(u_{n-1},u_{n-1},u_n)], \text{ where } \lambda = \frac{\gamma}{1-\gamma} < 1,$$

Thus, from Lemma 2.7 we say that sequence $\{u_n\}$ is a Cauchy sequence.

As (X, S) be a complete S-metric space, it follows that sequence $\{u_n\}$ is convergent to some point $w_0 \in X$, such that $u_n \to w_0$ as $n \to \infty$ or $S(u_n,u_n,w_0) \to 0$ as $n \to \infty$.

Now we shall show that w_0 is the common fixed point of T_1 and T_2.

Thus, by using triangular inequality and (3.1), we get

$$\begin{aligned} S(w_0,w_0,T_1w_0) &\leq S(w_0,w_0,u_{n+2}) + S(w_0,w_0,u_{n+2}) + S(T_1w_0,u_{n+2},u_{n+2}) \\ &= 2S(w_0,w_0,u_{n+2}) + S(T_1w_0,T_1w_0,T_2u_{n+1}) \\ &= 2S(w_0,w_0,u_{n+2}) + \gamma[\, S(w_0,w_0,T_1w_0) + S(u_{n+1},u_{n+1},T_2u_{n+1})] \\ &= \frac{1}{1-\gamma}[2S(w_0,w_0,u_{n+2}) + \gamma\, S(u_{n+1},u_{n+1},u_{n+2})]. \qquad (3.4) \end{aligned}$$

Taking limit on both sides of (3.4) as $n \to \infty$ and using continuity of S-metric function we get

$$\lim_{n\to\infty} S(w_0,w_0,T_1w_0)=0 \Longrightarrow T_1w_0 = w_0 .$$

Hence, w_0 is fixed point of T_1.

Similarly,

$$\begin{aligned} S(w_0,w_0,T_2w_0) &\leq 2S(w_0,w_0,u_{n+1}) + S(u_{n+1},u_{n+1},T_2w_0) \\ &= 2S(w_0,w_0,u_{n+1}) + S(T_1u_n,T_1u_n,T_2w_0) \\ &\leq 2S(w_0,w_0,u_{n+1}) + \gamma[\, S(u_n,u_n,T_1u_n) + S(w_0,w_0,T_2w_0)] \\ &= \frac{1}{1-\gamma}[2S(w_0,w_0,u_{n+1}) + \gamma\, S(u_n,u_n,T_1u_n)]. \qquad (3.5) \end{aligned}$$

Taking limit on both sides of (3.5) as $n \to \infty$ and using continuity of S-metric function we get

$$\lim_{n\to\infty} S(w_0,w_0,T_2w_0)=0 \Longrightarrow T_2w_0 = w_0 .$$

Hence, w_0 is fixed point of T_2 also.

To prove the uniqueness of the fixed point lets us suppose that $w_1 \in X$ be another fixed common point of T_1 and T_2, then $T_1w_1 = T_2w_1 = w_1$ and $w_0 \neq w_1$.

$$\begin{aligned} S(w_1,w_1,w_0) &= S(T_1w_1,T_1w_1,T_2w_0) \\ &\leq \gamma[S(w_1,w_1,T_1w_1) + S(w_0,w_0,T_2w_0)] \\ &= 0. \end{aligned}$$

Which implies $w_1 = w_0$, which is contradiction to the fact that two fixed point are distinct.

Hence, the common fixed point of T_1 and T_2 are unique.

Theorem 3.2. Let (X, S) be a complete S-metric space and let $T_1,T_2: X \to X$ be a pair of self maps such that

$$S(T_1u,T_1u,T_2v) \leq \alpha[S(u,u,T_1u)] + \beta[S(v,v,T_2v)], \qquad (3.6)$$

For all $u, v \in X$ where $\alpha, \beta \in [0,\frac{1}{2})$ along with $2\alpha + \beta < 1$. Then T_1 and T_2 has a unique common fixed point in X.

Proof: Let $u_0 \in X$ and we define a sequence $\{u_n\}$ such that $u_{2n+1} = T_1u_{2n}$ and $u_{2n+2} = T_2u_{2n+1}$ for $n = 0,1,2,3\ldots$

If there is $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then from (3.6)we have

$$S(u_{n_0+1},u_{n_0+1},u_{n_0+2})=S(T_1u_{n_0},T_1u_{n_0},T_2u_{n_{0+1}})$$
$$\leq\alpha[S(u_{n_0},u_{n_0},T_1u_{n_0})]+\beta[S(u_{n_{0+1}},u_{n_{0+1}},T_2u_{n_{0+1}})]$$
$$=\alpha[S(u_{n_0},u_{n_0},u_{n_{0+1}})]+\beta[S(u_{n_{0+1}},u_{n_{0+1}},u_{n_{0+2}})]$$
$$=\alpha[S(u_{n_0},u_{n_0},u_{n_0})]+\beta[S(u_{n_{0+1}},u_{n_{0+1}},u_{n_{0+2}})]$$
$$=\beta[S(u_{n_{0+1}},u_{n_{0+1}},u_{n_{0+2}})],$$

a contradiction, since $\beta<\frac{1}{2}<1$.

Thus it is permissible to assume that $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, the following cases are investigated.

Case I: n is even, Here $n = 2i$ for some $i = \{0,1,2,3\ldots\}$. From Equation (3.6) we obtain

$$S(u_{2i},u_{2i},u_{2i+1})=S(T_1u_{2i-1},T_1u_{2i-1},T_2u_{2i})$$
$$\leq\alpha[S(u_{2i-1},u_{2i-1},T_1u_{2i-1})]+\beta[S(u_{2i},u_{2i},T_2u_{2i})]$$
$$=\alpha[S(u_{2i-1},u_{2i-1},u_{2i})]+\beta[S(u_{2i},u_{2i},u_{2i+1})]$$
$$\leq\frac{\alpha}{1-\beta}[S(u_{2i-1},u_{2i-1},u_{2i})]. \tag{3.7}$$

Case II: n is odd, Here $n = 2i + 1$ for some $i = \{0,1,2,3\ldots\}$. Using simillar arguments given in case I, we have

$$S(u_{2i+1},u_{2i+1},u_{2i+2})\leq\frac{\alpha}{1-\beta}[S(u_{2i},u_{2i},u_{2i+1})]. \tag{3.8}$$

Combining equation (3.7) and (3.8) together, we obtained

$$S(u_n,u_n,u_{n+1})\leq\frac{\alpha}{1-\beta}[S(u_{n-1},u_{n-1},u_n)],$$

Thus,

$$S(u_n,u_n,u_{n+1})\leq\lambda[S(u_{n-1},u_{n-1},u_n)], \text{ where } \lambda=\frac{\alpha}{1-\beta},$$

Since, $2\alpha+\beta<1$which implies that

$$2\alpha<1-\beta \Rightarrow \frac{\alpha}{1-\beta}<\frac{1}{2}<1,$$

So, we get $\lambda<1$, and hence from Lemma 2.7 we say that sequence $\{u_n\}$ is a Cauchy sequence.

As (X, S) be a complete S-metric space, it follows that sequence $\{u_n\}$ is convergent to some point $w_0 \in X$, such that $u_n \to w_0$ as $n\to\infty$ or $S(u_n,u_n,w_0)\to 0$ as $n\to\infty$.

Now we shall show that w_0 is the common fixed point of T_1 and T_2.

Thus, by using triangular inequality and (3.6), we get

$$S(w_0,w_0,T_1w_0)\leq 2S(w_0,w_0,u_{n+2})+S(u_{n+2},u_{n+2},T_1w_0)$$
$$=2S(w_0,w_0,u_{n+2})+S(T_2u_{n+1},T_2u_{n+1},T_1w_0)$$
$$\leq 2S(w_0,w_0,u_{n+2})+S(T_2u_{n+1},T_2u_{n+1},T_1w_0)$$
$$=2S(w_0,w_0,u_{n+2})+S(T_1w_0,T_1w_0,T_2u_{n+1})$$
$$\leq 2S(w_0,w_0,u_{n+2})+\alpha S(w_0,w_0,T_1w_0)+\beta S(u_{n+1},u_{n+1},T_2u_{n+1})$$
$$=2S(w_0,w_0,u_{n+2})+\alpha S(w_0,w_0,T_1w_0)+\beta S(u_{n+1},u_{n+1},u_{n+2})$$
$$\leq\frac{1}{1-\alpha}[2S(w_0,w_0,u_{n+2})+\beta S(u_{n+1},u_{n+1},u_{n+2})]. \tag{3.9}$$

Taking limit on both sides of (3.4) as $n\to\infty$ and using continuity of S-metric function we get

$$\lim_{n\to\infty} S(w_0, w_0, T_1w_0) = 0 \Longrightarrow T_1w_0 = w_0.$$

Hence, w_0 is fixed point of T_1.

Similarly,

$$\begin{aligned} S(w_0, w_0, T_2w_0) &\leq 2S(w_0, w_0, u_{n+1}) + S(u_{n+1}, u_{n+1}, T_2w_0) \\ &= 2S(w_0, w_0, u_{n+1}) + S(T_1u_n, T_1u_n, T_2w_0) \\ &\leq 2S(w_0, w_0, u_{n+1}) + \alpha S(u_n, u_n, T_1u_n) + \beta S(w_0, w_0, T_2w_0) \\ &= 2S(w_0, w_0, u_{n+1}) + S(T_1w_0, T_1w_0, T_2u_{n+1}) \\ &\leq \frac{1}{1-\beta}[2S(w_0, w_0, u_{n+1}) + \alpha S(u_n, u_n, T_1u_n)]. \end{aligned} \quad (3.10)$$

Taking limit on both sides of (3.5) as $n \to \infty$ and using continuity of S-metric function we get

$$\lim_{n\to\infty} S(w_0, w_0, T_2w_0) = 0 \Longrightarrow T_2w_0 = w_0.$$

Hence, w_0 is fixed point of T_2 also.

Further we shall show that the fixed point for the two mappings we obtained are unique.

Suppose $w_1 \in X$ be another fixed common point of T_1 and T_2, then $T_1w_1 = T_2w_1 = w_1$ and $w_0 \neq w_1$.

$$\begin{aligned} S(T_1w_1, T_1w_1, T_2w_0) &\leq \alpha S(w_1, w_1, T_1w_1) + \beta S(w_0, w_0, T_2w_0) \\ &\leq \alpha S(T_1w_1, T_1w_1, T_1w_1) + \beta S(T_2w_0, T_2w_0, T_2w_0) \\ &= 0. \end{aligned}$$

Which implies $T_1w_1 = T_2w_0$ and hence $w_1 = w_0$, which is contradiction to the fact that two fixed point are distinct.

Hence, the common fixed point of T_1 and T_2 are unique.

Note 3.3: Theorem 3.2 uses a generalization of the Kannan contraction function (if $\alpha = \beta$ then Equation (3.1) becomes the Kannan contraction function of Theorem 3.1).

Example 3.4: Let (X, S) be a complete S-metric space define by

$$S(u,v,w) = (u-v)^2 + (v-w)^2 + (w-u)^2$$

Let $X = [0,1] \subset \mathbb{R}$ and we define self mappings $T_1, T_2: X \to X$ as

$$T_1u = \begin{cases} \frac{u}{3}, & u \in \left[0, \frac{1}{2}\right] \\ \frac{u}{5}, & u \in \left[\frac{1}{2}, 1\right] \end{cases} \text{ and } T_1u = \begin{cases} \frac{u}{4}, & u \in \left[0, \frac{1}{3}\right] \\ \frac{u}{6}, & u \in \left[\frac{1}{3}, 1\right] \end{cases},$$

Then the condition

$S(T_1u, T_1u, T_2v) \leq \alpha[S(u,u,T_1u)] + \beta[S(v,v,T_2v)]$,

is satisfied for all $u, v \in X$ by taking $\alpha = \beta = \frac{1}{4}$ and is also satisfies that $2\alpha + \beta = \frac{3}{4} < 1$.

Thus, by Theorem 3.2 T_1 and T_2 has a unique common fixed point in X.

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