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SOME NEW FINDINGS IN DIVIDED SQUARE DIFFERENCE CORDIAL GRAPHS

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Abstract : In this paper I have proved four new results admitting Divided Square Difference Cordial Labeling.

1. Every $\mathrm{T_p}$ − Tree is a divided square difference cordial graph.
2. The middle graph $\mathrm{M}(P_n)$ is a divided square difference cordial graph.
3. The splitting graph $S'(P_n)$ is a divided square difference cordial graph.
4. The splitting graph $S'(C_n)$ is a divided square difference cordial graph, if $n \equiv 0, 1, 3 (mod\ 4)$.

Keywords: Divided square difference cordial labeling, degree splitting, Middle graph.
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1. INTRODUCTION

Let G = (V, E) be a simple, finite, undirected and non-trivial graph with the vertex set V. The number of elements of V, denoted as |V(G)| is called the order of G while the number of elements of E, denoted as |E(G)| is called the size of G. More detail of graph labeling results and its applications can be found in Gallian [2]. I provide brief summary of definitions and other related information which are useful for the further investigations.

The concept of divided square difference cordial labeling was given by A. Alfred Leo, R. Vikramaprasad and R. Dhavaseelan[4]. They proved that path, cycle, Star graph, K2,n are divided square difference cordial graphs.

Definition 1. [1] Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let there be two pendant vertices u and v in T such that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. If the edge $u_0 v_0$ is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an ept) and the edge $u_0 v_0$ is called transformable edge.

If T can be reduced to a path by the sequence of ept's, then T is called a T_P – tree (transformed tree) and such a sequence regarded as a composition of mappings (ept's) denoted by P, is called a parallel transformation of T. The path, the image of T under P is denoted by P(T).

A T_P – tree T **An ept $P_1(T)$** **Second ept $P_2(T)$**

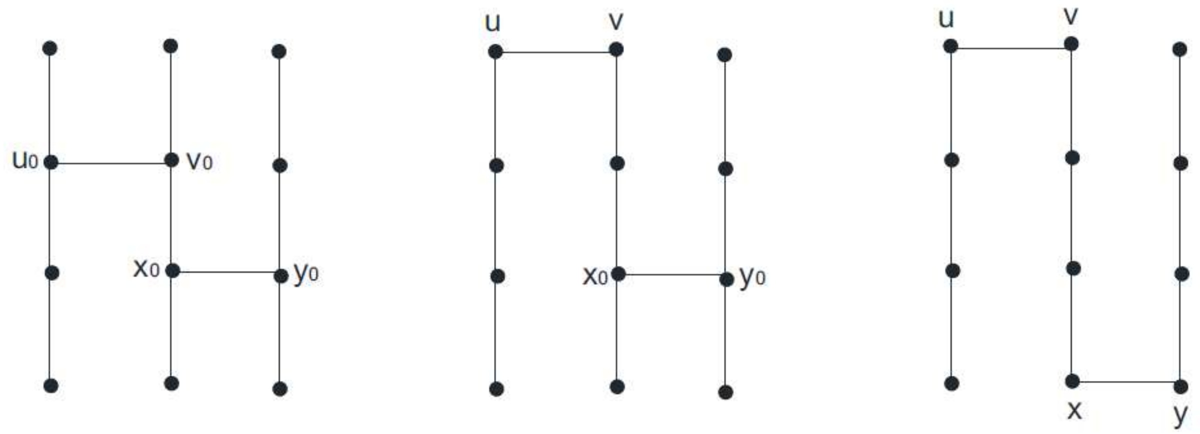

Figure 1
A T_P -tree and a sequence of two ept's reducing it to a path are shown in Figure 1.

Definition 2. [2] For a graph G the splitting graph S′ (G) of a graph G is obtained by adding a new vertex v ′ corresponding to each vertex v of G such that N(v) = N(v′).

Definition 3. [3] The middle graph M(G) of a graph G is the graph whose vertex set is V(G) ∪ E(G) and in which two vertices are adjacent if and only if either theyare adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

2. MAIN RESULTS

Theorem 1. Every T_p − Tree is a divided square difference cordial graph.
Proof. Let T be a T_p − Tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path P(T), i have
(I) $V(P(T)) = V(T)$ and
(II) $E(P(T)) = (E(T) - E_d) \cup E_p$,
Where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequences $P = (P_1, P_2, \dots, P_r)$ of the P used to arrive at the path P(T). Clearly, E_d and E_p have the same number of edges. Denote the vertices of P(T) successively as $u_1, u_2, \dots, u_m$ starting from one pendant vertex of P(T) right up to the other.
Define $g: V(T) \to \{1,2,\dots,m\}$ as follows.
Case 1: m is odd and $1 \le j \le m$.

$$G(u_j) = \begin{cases} j; & \text{if } j \equiv 0,1 \pmod 4 \\ j+1; & \text{if } j \equiv 2 \pmod 4. \\ j-1; & \text{if } j \equiv 3 \pmod 4 \end{cases}$$

Let $u_j u_k$ be a transformed edge in T, $1 \le j < k \le m$ and let P_1 be the ept obtained by deleting the edge$u_j u_k$ and adding the edge $u_{j+t} u_{k-t}$ where t is the distance of u_j from u_{j+t} and the distance of u_k from u_{k-t}. Let P be a parallel transformation of T that contains P1 as one of the constituent epts.
Since $u_{j+t} u_{k-t}$ is an edge in the path P(T), it follows that j + t + 1 = k – t which implies that k = j + 2t + 1. Therefore, j and k are of opposite parity.
The induced edge label $u_j u_k$ is given by,

$$g^*(u_j u_k) = g^*(u_j u_{j+2t+1}) = \begin{cases} 1; & \text{if } j \equiv 0,2 \pmod 4 \text{ and } 1 \le j \le m \\ 0; & \text{f } j \equiv 1,3 \pmod 4 \text{ and } 1 \le j \le m. \end{cases}$$

The induced edge label $u_{j+t} u_{k-t}$ is given by,

$$g^*(u_{j+t} u_{k-t}) = g^*(u_{j+t} u_{j+t+1}) = \begin{cases} 1; & \text{if } j \equiv 0,2 \pmod 4 \text{ and } 1 \le j \le m \\ 0; & \text{f } j \equiv 1,3 \pmod 4 \text{ and } 1 \le j \le m. \end{cases}$$

Therefore $g^*(u_j u_k) = g^*(u_{j+t} u_{k-t})$.
The induced edge labels are,

$$g^*(u_j u_{j+1}) = \begin{cases} 1; & \text{if } j \text{ is even and } 1 \le j \le m-1 \\ 0; & \text{if } j \text{ is odd and } 1 \le j \le m-1. \end{cases}$$

Case 2: m is even and $1 \le j \le m$.

$$G(u_j) = \begin{cases} j; & \text{if } j \equiv 1,2 \pmod 4 \\ j+1; & \text{if } j \equiv 3 \pmod 4. \\ j-1; & \text{if } j \equiv 0 \pmod 4 \end{cases}$$

Let $u_j u_k$ be a transformed edge in T, $1 \le j < k \le m$ and let P_1 be the ept obtained by deleting the edge$u_j u_k$ and adding the edge $u_{j+t} u_{k-t}$ where t is the distance of u_j from u_{j+t} and the distance of u_k from u_{k-t}. Let P be a parallel transformation of T that contains P1 as one of the constituent epts.
Since $u_{j+t} u_{k-t}$ is an edge in the path P(T), it follows that j + t + 1 = k – t which implies that k = j + 2t + 1. Therefore, j and k are of opposite parity.
The induced edge label $u_j u_k$ is given by,

$$g^*(u_j u_k) = g^*(u_j u_{j+2t+1}) = \begin{cases} 1; & \text{if } j \equiv 1,3 \pmod 4 \text{ and } 1 \le j \le m \\ 0; & \text{f } j \equiv 0,2 \pmod 4 \text{ and } 1 \le j \le m. \end{cases}$$

The induced edge label $u_{j+t} u_{k-t}$ is given by,

$$g^*(u_{j+t} u_{k-t}) = g^*(u_{j+t} u_{j+t+1}) = \begin{cases} 1; & \text{if } j \equiv 1,3 \pmod 4 \text{ and } 1 \le j \le m \\ 0; & \text{f } j \equiv 0,2 \pmod 4 \text{ and } 1 \le j \le m. \end{cases}$$

Therefore $g^*(u_j u_k) = g^*(u_{j+t} u_{k-t})$.
The induced edge labels are,

$$g^*(u_j u_{j+1}) = \begin{cases} 1; & \text{if } j \text{ is odd and } 1 \le j \le m-1 \\ 0; & \text{if } j \text{ is even and } 1 \le j \le m-1. \end{cases}$$

In the above two case, observe that $|e_g(0) - e_g(1)| \le 1$.
Hence, T_p − tree T admits a divided square difference cordial labelling.

Theorem 2. The middle graph $M(P_n)$ is a divided square difference cordial graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n. Let V(M(P_n)) = $\{v_i: 1 \leq i \leq n\} \cup \{v'_i: 1 \leq i \leq n-1\}$ and E(M(P_n)) = $\{v_i v'_i, v'_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v'_i v'_{i+1}: 1 \leq i \leq n-2\}$. Therefore M($P_n$) is of order 2n – 1 and size 3n – 4.
Define $g: V(\mathrm{M}(P_n)) \rightarrow \{1,2, \ldots, 2n-1\}$ as follows.

Case 1: n is odd.

$$g(v_i) = \begin{cases} 2i; & \text{if } i \equiv 2 \pmod 4 \text{ and } 1 \leq i \leq n, \\ 2i-1; & \text{if } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq i \leq n, \\ 2i-2; & \text{if } i \equiv 0 \pmod 4 \text{ and } 1 \leq i \leq n. \end{cases}$$

$$g(v'_i) = \begin{cases} 2i; & \text{if } i \equiv 0,1 \pmod 4 \text{ and } 1 \leq i \leq n-1, \\ 2i-1; & \text{if } i \equiv 2 \pmod 4 \text{ and } 1 \leq i \leq n-1, \\ 2i+1; & \text{if } i \equiv 3 \pmod 4 \text{ and } 1 \leq i \leq n-1. \end{cases}$$

Then the induced edge labels are as follows:

$$g^*(v_i v'_i) = \begin{cases} 1; & \text{if } i \equiv 1,2 \pmod 4 \text{ and } 1 \leq i \leq n, \\ 0; & \text{if } i \equiv 0,3 \pmod 4 \text{ and } 1 \leq i \leq n. \end{cases}$$

$$g^*(v_{i+1} v'_i) = \begin{cases} 1; & \text{if } i \equiv 0,3 \pmod 4 \text{ and } 1 \leq i \leq n-1, \\ 0; & \text{if } i \equiv 1,2 \pmod 4 \text{ and } 1 \leq i \leq n-1. \end{cases}$$

$$g^*(v'_{i+1} v'_i) = \begin{cases} 1; & \text{if } i \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n-2, \\ 0; & \text{if } i \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq n-2. \end{cases}$$

Case 2: n is even.

$$g(v_i) = \begin{cases} 1; & \text{if } i = 1, \\ 2i; & \text{if } i \equiv 3 \pmod 4 \text{ and } 2 \leq i \leq n, \\ 2i-1; & \text{if } i \equiv 0 \pmod 4 \text{ and } 2 \leq i \leq n, \\ 2i-2; & \text{if } i \equiv 1,2 \pmod 4 \text{ and } 2 \leq i \leq n. \end{cases}$$

$$g(v'_i) = \begin{cases} 2i; & \text{if } i \equiv 2 \pmod 4 \text{ and } 1 \leq i \leq n-1, \\ 2i-1; & \text{if } i \equiv 3 \pmod 4 \text{ and } 1 \leq i \leq n-1, \\ 2i+1; & \text{if } i \equiv 0,1 \pmod 4 \text{ and } 1 \leq i \leq n-1. \end{cases}$$

Then the induced edge labels are as follows:

$$g^*(v_1 v'_1) = 1$$

$$g^*(v_i v'_i) = \begin{cases} 1; & \text{if } i \equiv 1 \pmod 2 \text{ and } 2 \leq i \leq n, \\ 0; & \text{if } i \equiv 0 \pmod 2 \text{ and } 2 \leq i \leq n. \end{cases}$$

$$g^*(v_{i+1} v'_i) = \begin{cases} 1; & \text{if } i \equiv 0,1 \pmod 4 \text{ and } 1 \leq i \leq n-1, \\ 0; & \text{if } i \equiv 2,3 \pmod 4 \text{ and } 1 \leq i \leq n-1. \end{cases}$$

$$g^*(v'_{i+1} v'_i) = \begin{cases} 1; & \text{if } i \equiv 1,2 \pmod 4 \text{ and } 1 \leq i \leq n-2, \\ 0; & \text{if } i \equiv 0,3 \pmod 4 \text{ and } 1 \leq i \leq n-2. \end{cases}$$

In the above two cases, observe that $e_g(1) = \left\lceil \frac{3n-4}{2} \right\rceil, e_g(0) = \left\lfloor \frac{3n-4}{2} \right\rfloor$.
Therefore, $|e_g(0) - e_g(1)| \leq 1$.
Hence, M(P_n) is a divided square difference cordial graph.

Theorem 3. The splitting graph $S'(P_n)$ is a divided square difference cordial graph.
Proof. Let Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n. Let V($S'(P_n)$) = $\{v_i, v'_i: 1 \leq i \leq n\}$ and E($S'(P_n)$) = $\{v_i v'_{i+1}, v'_i v_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\}$. Therefore $S'(P_n)$ is of order 2n and size 3n – 3.
Define $g: V(S'(P_n)) \rightarrow \{1,2, \ldots, 2n\}$ as follows.

$$g(v_i) = \begin{cases} 2i; & \text{if } i \equiv 0,3 \pmod 4 \text{ and } 2 \leq i \leq n, \\ 2i-1; & \text{if } i \equiv 1,2 \pmod 4 \text{ and } 2 \leq i \leq n. \end{cases}$$

$$g(v'_i) = \begin{cases} 2i; & \text{if } i \equiv 1,2 \pmod 4 \text{ and } 1 \leq i \leq n-1, \\ 2i-1; & \text{if } i \equiv 0,3 \pmod 4 \text{ and } 1 \leq i \leq n-1. \end{cases}$$

Then the induced edge labels are as follows:

$$g^*(v_i v'_{i+1}) = \begin{cases} 1; & \text{if } i \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n-1, \\ 0; & \text{if } i \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq n-1. \end{cases}$$

$$g^*(v_{i+1} v'_i) = \begin{cases} 1; & \text{if } i \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n-1, \\ 0; & \text{if } i \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq n-1. \end{cases}$$

$$g^*(v_{i+1} v_i) = \begin{cases} 1; & \text{if } i \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq n-1, \\ 0; & \text{if } i \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n-1. \end{cases}$$

In the above two cases, observe that $e_g(1) = \left\lceil \frac{3n-3}{2} \right\rceil, e_g(0) = \left\lfloor \frac{3n-3}{2} \right\rfloor$.
Therefore, $|e_g(0) - e_g(1)| \leq 1$.
Hence, splitting graph $S'(P_n)$ is a divided square difference cordial graph.

Theorem 4. The splitting graph $S'(C_n)$ is a divided square difference cordial graph, if $n \equiv 0, 1, 3(mod\ 4)$.
Proof. Let $v'_1, v'_2, \dots, v'_n$ be the added vertices corresponding to $v_1, v_2, \dots, v_n$ of the cycle C_n.
Also, $S'(C_n)$ is of size 3n and order 2n.
Define $g: V(S'(C_n)) \to \{1,2, \dots, 2n\}$ as follows.
Case 1: for n = 3.
$g(v_1) = 1$, $g(v_2) = 3$, $g(v_3) = 2$, $g(v'_1) = 4$, $g(v'_2) = 6$, $g(v'_3) = 5$.
Then the induced edge labels are $g^*(v_2v_3) = g^*(v_3v_1) = g^*(v'_2v_1) = g^*(v'_1v_2) = 1$ and $g^*(v_1v_2) = g^*(v_3v'_1) = g^*(v'_3v_1) = g^*(v'_3v_2) = g^*(v'_2v_3) = 0$.
Therefore $e_g(0) = 5$, $e_g(1) = 4$.
Case 2: for n > 3.
Subcase 1. $\boldsymbol{n \equiv 0(mod\ 4)}$
For $1 \le i \le n$.

$$g(v_i) = \begin{cases} i-1; & i \equiv 3(mod\ 4) \\ i; & i \equiv 0, 1(mod\ 4) \\ i+1; & i \equiv 2(mod\ 4) \end{cases}$$

$$g(v'_i) = g(v_n) + i.$$

Then the induced edge labels are as follows:
For $1 \le i \le n-1$.

$$g^*(v_iv_{i+1}) = \begin{cases} 1; & \text{if i is even;} \\ 0; & \text{if i is odd.} \end{cases}$$

$$g^*(v_{i+1}v'_i) = \begin{cases} 1; & \text{if } i \equiv 0, 3(\text{mod } 4), \\ 0; & \text{if } i \equiv 1, 2(\text{mod } 4). \end{cases}$$

$$g^*(v_1v'_n) = g^*(v'_1v_n) = g^*(v_1v_n) = 1;$$

$$g^*(v'_{i+1}v_i) = \begin{cases} 1; & \text{if } i \equiv 0, 1(\text{mod } 4), \\ 0; & \text{if } i \equiv 2, 3(\text{mod } 4). \end{cases}$$

Therefore $e_g(0) = e_g(1) = \frac{3n}{2}$.
Subcase 2. $\boldsymbol{n \equiv 1, 3(mod\ 4)}$**.**
Assign the labels to the vertices v_i, v'_i for For $1 \le i \le n$ as in subcaase 1.
Then the induced edge labels are as follows:
For $1 \le i \le n-1$.

$$g^*(v_iv_{i+1}) = \begin{cases} 1; & \text{if i is even;} \\ 0; & \text{if i is odd.} \end{cases}$$

$$g^*(v_nv_1) = 1$$

$$g^*(v_{i+1}v'_i) = \begin{cases} 1; & \text{if } i \equiv 1, 2(\text{mod } 4), \\ 0; & \text{if } i \equiv 0, 3(\text{mod } 4). \end{cases}$$

$$g^*(v_1v'_n) = g^*(v'_1v_n) = 0;$$

$$g^*(v'_{i+1}v_i) = \begin{cases} 1; & \text{if } i \equiv 2, 3(\text{mod } 4), \\ 0; & \text{if } i \equiv 0, 1(\text{mod } 4). \end{cases}$$

Therefore $e_g(0) = \left\lfloor \frac{3n}{2} \right\rfloor$, $e_g(1) = \left\lceil \frac{3n}{2} \right\rceil$.
From the above two cases $|e_g(0) - e_g(1)| \le 1$.
Hence, the splitting graph $S'(C_n)$ is a divided square difference cordial graph, if $n \equiv 0,1,3(mod\ 4)$.

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