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# $\phi$-Primary Subtractive ideals in Ternary Semirings

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***Abstract:*** Let $S$ be a commutative ternary semiring. In this paper I have defined $\phi$-primary ideal in $S$ and prove the result $I$ is $\phi$-primary ideal if and only if for ideals $A, B$ and $C$ of $S, ABC \subseteq I - \phi(I)$ implies $A \subseteq I$ or $B \subseteq I$ or $C \subseteq \sqrt{I}$, where $I$ and $\phi(I)$ are subtractive ideals of $S$.

**Keywords:** ternary semiring, $\phi$-primary ideal, subtractive ideal, $\phi$-semiprime ternary semiring

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## I. Introduction

The concept of primary ideals and its extension play a significant role in for their uses in a variety of fields, including graph theory, commutative algebra topological spaces, algebraic geometry, information science, coding theory, etc. Here we generalize the concept of $\phi$-primary ideal from semirings [5] to ternary semirings. Here our approach is purely algebraic.

**Definition 1.1.** [2] A non-empty set $S$ together with a binary operation called addition $(+)$ and a ternary operation called ternary multiplication $(\cdot)$ is called ternary semiring if $S$ is an additive commutative semigroup satisfying the following conditions:

i) $(a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e)$;

ii) there exists $0 \in S$ such that $a + 0 = a = 0 + a, a \cdot b \cdot 0 = a \cdot 0 \cdot b = 0 \cdot a \cdot b = 0$;

iii) $(a + b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d, a \cdot (b + c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d,$
$a \cdot b \cdot (c + d) = a \cdot b \cdot c + a \cdot b \cdot d;$ for all $a, b, c, d, e \in S$.

**Definition 1.2.** [2] A ternary semiring $S$ is said to be commutative if,

$a \cdot b \cdot c = a \cdot c \cdot b = c \cdot a \cdot b$ for all $a, b, c \in S$.

**Note.** For convenience we write $ab$ instead of $a \cdot b$.

**Definition 1.3.** [2] A subset $K$ of a ternary semiring $S$ is called a ternary sub-semiring if $K$ is itself a ternary semiring under the same operations on $S$.

**Definition 1.4.** [2] A non-empty subset $I$ of a ternary semiring $S$ is called a left (resp. lateral, right) ideal of $S$ if the following conditions are satisfied:

i) $a, b \in I$ implies $a + b \in I$;

ii) $a \in I,\ r, s \in S$ implies $rsa$ (resp. $ras, ars$) $\in I$.

If $I$ is a left, a lateral and a right ideal of $S$, then $I$ is called an ideal of $S$.

**Definition.** [2] An ideal $I$ of a ternary semiring $S$ is called subtractive ideal ($k$-ideal) if for $a, a + b \in I, b \in S$ implies $b \in S$.

**Definition.** [1] An ideal $I \neq S$ of a ternary semiring $S$ is called prime if $abc \in I$, where $a, b, c \in S$ implies $a \in I$ or $b \in I$ or $c \in I$.

## II. $\phi$-PRIMARY SUBTRACTIVE IDEALS

**Definition 2.1.** Let $S$ be commutative ternary semiring. Let $I(S)$ be the set of all ideals of $S$ and $I^*(S)$ be set of all proper ideals of $S$ with a function, $\phi: I(S) \to I^*(S)$. A proper ideal $I$ of $S$ is called a $\phi$-primary ideal if for all $a, b, c \in S$, with $abc \in I - \phi(I)$ implies $a \in I$ or $b \in I$ or $c^n \in I$ for some odd positive integer $n$.

Note. Since $I - \phi(I) = I - (I \cap \phi(I))$, we have $\phi(I) \subseteq I$.

**Lemma 2.2.** Every prime ideal in a commutative ternary semiring $S$ is $\phi$-primary.

**Lemma 2.3.** Every $\phi$-prime ideal of a commutative ternary semiring $S$ is $\phi$-primary.

**Lemma 2.4.** If $I$ is a $\phi$-primary ideal of a commutative ternary semiring $S$ and $\phi(I)$ is a primary ideal, then $I$ is a primary ideal of $S$.

**Proof.** Let $a, b, c \in S$. Let $I$ be an $\phi$-primary ideal of $S$. Suppose that $abc \in I$, if $abc \notin \phi(I)$ as $I$ is $\phi$-primary ideal, $a \in I$ or $b \in I$ or $c^n \in I$ for some odd positive integer $n$ and if $abc \in \phi(I)$, as $\phi(I)$ is a primary ideal, $a \in \phi(I) \subseteq I$ or $b \in \phi(I) \subseteq I$ or $c^n \in \phi(I) \subseteq I$, for some odd positive integer $n$.

**Definition 2.5.** Given two functions $\psi_1, \psi_2 : I(S) \to I(S) \cup \{\Phi\}$. We define $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for each $J \in I(S)$.

**Proposition 2.6.** Let $S$ be a commutative ternary semiring and $J$ be proper ideal of $S$. Let $\psi_1, \psi_2 : I(S) \to I(S) \cup \{\Phi\}$ be functions with $\psi_1 \leq \psi_2$. Then if $J$ is $\psi_1$-primary ideal, then $J$ is $\psi_2$-primary ideal.

**Proof.** Suppose that, $abc \in J - \psi_2(I)$, where $a, b, c \in S$. As $\psi_1 \leq \psi_2$, we have $\psi_1(J) \subseteq \psi_2(J)$. Hence $abc \in J - \psi_1(I)$ and therefore $a \in J$ or $b \in J$ or $c^n \in J$ for some odd positive integer $n$. Thus $J$ is $\psi_2$-primary ideal.

**Theorem 2.7.** Let $I$ be a $\phi$-primary subtractive ideal of a commutative ternary semiring $S$. Then $I^3 \subseteq \phi(I)$ or $I$ is a primary ideal.

**Proof.** Suppose that $I^3 \nsubseteq \phi(I)$. Let $abc \in I$ for $a, b, c \in S$.

If $abc \notin \phi(I)$, then $a \in I$ or $b \in I$ or $c^n \in I$ for some odd positive integer $n$. So, in this case $I$ is primary ideal.

Assume that $abc \in \phi(I)$. If $abI \nsubseteq \phi(I)$, then there exists $z \in I$ such that $abz \notin \phi(I)$.

Now, $abc + abz \notin \phi(I)$, that is $ab(z + c) \notin \phi(I)$. Hence $a \in I$ or $b \in I$ or $(c + z)^n \in I$, for some odd positive integer $n$. Since $(c + z)^n = \sum_{k=0}^{n} c^k z^{n-k} \in I$ and since $z \in I$ we have $c^n \in I$. So $a \in I$ or $b \in I$ or $c^n \in I$.

Now suppose that $abI \subseteq \phi(I)$. If $aIc \nsubseteq \phi(I)$, then there exists $y \in I$ such that $ayc \notin \phi(I)$. Now $a(b + y)c \in I - \phi(I)$ implies $a \in I$ or $b + y \in I$ or $c^n \in I$. As $I$ is subtractive $b \in I$. So $a \in I$ or $b \in I$ or $c^n \in I$. Hence suppose that $aIc \subseteq \phi(I)$. Similarly for other cases we get same result. Since $I^3 \nsubseteq \phi(I)$.

There exists $r, s, t \in I$ such that $rst \notin \phi(I)$, then $(a + r)(b + s)(c + t) \in I - \phi(I)$. So $a \in I$ or $b \in I$ or $c^n \in I$. Hence $I$ is primary ideal.

**Definition 2.8.** A commutative ternary semiring $S$ is called $\phi$-semiprime if $I^n \subseteq \phi(I)$ for each ideal $I$ of $S$ and an odd positive integer $n$, implies $I \subseteq \phi(I)$.

**Corollary 2.9.** Let $S$ be a $\phi$-semiprime commutative ternary semiring. Then a subtractive ideal $J$ of $S$ is $\phi$-primary if and only if $I = \phi(I)$ or $I$ is primary.

Proof. Let $n$ be an odd positive integer. If $n = 1, I = \phi(I)$.

Suppose that $n > 1$, we have $I^n = I^3 . I^{n-3}$. By Theorem 2.7 and definition 2.8 result holds trivially.

**Definition 2.10.** Let $I$ be an ideal in a commutative ternary semiring $S$ and let

$\sqrt{I} = \{x \in S : x^m \in I, for\ some\ odd\ positive\ integer\ m\}$, then $\sqrt{I}$ is an ideal of $S$ and $I \subseteq \sqrt{I}$.

**Lemma 2.11.** Let $I$ be primary ideal of a commutative ternary semiring $S$. Then $\sqrt{I}$ is a prime ideal of $S$.

Proof. Let $abc \in \sqrt{I}$, where $a, b, c \in S$. Then there exists odd positive integer $m$ such that $(abc)^m = a^m b^m c^m \in I$. Since $I$ is primary ideal we have $a^m \in I$ or $b^m \in I$ or $(c^m)^n \in I$ for some odd positive integer $n$.

Thus $a \in \sqrt{I}$ or $b \in \sqrt{I}$ or $c \in \sqrt{I}$. Hence $\sqrt{I}$ is a prime ideal of $S$.

**Lemma 2.12.** Let $I$ be a $\phi$-primary ideal of a commutative ternary semiring $S$, with $\sqrt{\phi(I)} = \phi(\sqrt{I})$. Then $\sqrt{I}$ is $\phi$-prime ideal of $S$.

**Proof.** Let $abc \in \sqrt{I} - \phi(\sqrt{I})$ and $a, b \notin \sqrt{I}$, where there exists an odd positive integer $m$ such that $a^m b^m c^m \in I$.

If $(abc)^m \in \phi(I)$, then $abc \in \sqrt{\phi(I)} = \phi(\sqrt{I})$ which is not possible. Hence $(abc)^m \notin \phi(I)$, so $(abc)^m \in I - \phi(I)$. But $I$ is $\phi$-primary ideal and $a, b \notin \sqrt{I}$ . hence $c \in \sqrt{I}$.

**Lemma 2.13.** If $I$ and $J$ be subtractive ideals of a ternary semiring $S$. Then $I \cup J = I$ or $I \cup J = J$.

**Lemma 2.14.[4]** Let $S$ be a commutative ternary semiring, $I$ be ideal of $S$ and $x, y \notin I$.

(i) If $I$ is subtractive, then $(I: x)$ is also subtractive.

(ii) If $I$ is prime ideal of $S$ and $(I: x)$ is subtractive, then $I$ is subtractive.

**Theorem 2.15.** Let $S$ be a commutative ternary semiring and $I$ a subtractive ideal of $S$, then the following statements are equivalent:

(i) $I$ is $\phi$-primary ideal.

(ii) If $x \notin \sqrt{I}$, then $(I: x) = I \cup (\phi(I): x)$.

(iii) If $x \notin \sqrt{I}$, then $(I: x) = I$ or $(I: x) = (\phi(I): x)$.

**Proof.** $(i) \Rightarrow (ii)$ Let $t \in (I: x)$, then $txy \in I$.

If $txy \in \phi(I)$ for some $y \notin I$, then $t \in (\phi(I): x) \subseteq I \cup (\phi(I): x)$.

If $txy \notin \phi(I)$ for some $y \notin I$, then $t^n \in I \subseteq I \cup (\phi(I): x)$. So, $(I: x) \subseteq I \cup (\phi(I): x)$.

On the other hand, let $k \in I \cup (\phi(I): x)$. Then $kxy \in \phi(I) \subseteq I$. Now $k \in (I: x)$.

$(ii) \Rightarrow (iii)$ Since $I$ is subtractive ideal by Lemma 2.13 and Lemma 2.15 we get the required result.

$(iii) \Rightarrow (i)$ Let $xyz \in I - \phi(I)$ and $x, z \notin \sqrt{I}$, then $y \notin (\phi(I): x)$ and $y \in (I : x)$ so $y \in I$.

**Lemma 2.16.** Let $I$ be a $\phi$-primary subtractive ideal of a commutative ternary semiring $S$ that is not primary, then $\sqrt{\phi(I)} = \sqrt{I}$.

**Proof.** Let $I$ be a a $\phi$-primary subtractive ideal of a commutative ternary semiring $S$ that is not primary, then by Theorem 2.7 $I^3 \subseteq \phi(I)$. So $I \subseteq \sqrt{\phi(I)}$ hence $\sqrt{I} \subseteq \sqrt{\phi(I)}$ . Obviously $\sqrt{\phi(I)} = \sqrt{I}$. Hence $\sqrt{\phi(I)} = \sqrt{I}$.

**Theorem 2.17.** Let $I$ and $\phi(I)$ are subtractive ideals of a commutative ternary semiring $S$. Then $I$ is $\phi$-primary ideal if and only if for ideals $A, B$ and $C$ of $S, ABC \subseteq I - \phi(I)$ implies $A \subseteq I$ or $B \subseteq I$ or $C \subseteq \sqrt{I}$.

**Proof.** Suppose that $I$ is a $\phi$-primary ideal of $S$. If $I$ is primary, then nothing to prove. So, suppose that $I$ is not primary ideal. Let $A, B$ and $C$ be ideals of $S$ such that $ABC \in I - \phi(I), A \nsubseteq I, B \nsubseteq I$ and $C \nsubseteq \sqrt{I}$. Let $b \in B, c \in C$. If $c \notin \sqrt{I}$, then either $(I: c) = I$ or $(I: c) = (\phi(I): c)$. Now $Abc \subseteq ABC \subseteq I$, where $A \subseteq (I: c)$. Choose $a \in A - I$, then $a \in (I: c) = I$, so, we get $A \nsubseteq I$ and $(I: c) = (\phi(I): c)$. Therefore $A \subseteq (\phi(I): c)$ and $Abc \subseteq \phi(I)$.

Now suppose that $c \in \sqrt{I}$, then $c \in I$ and $c \in \sqrt{I} \cap C$, choose $c' \in C - \sqrt{I}$, then $(c + c') \in C - \sqrt{I}$ and hence $Abc' \subseteq \phi(I)$ and $Ab(c + c') \in \phi(I)$. So $Abc \subseteq \phi(I)$. Therefore $ABC \in \phi(I)$. A contradiction.

Conversely, if $abc \in I - \phi(I)$, where $a, b, c \in S$, then , $< a >< b >< c > \subseteq I - \phi(I)$. Hence $< a > \subseteq I$ or $< b > \subseteq I$ or $< c > \subseteq \sqrt{I}$.

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