



EXISTENCE AND UNIQUENESS THEOREMS FOR FUZZY DIFFERENTIAL EQUATIONS

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Abstract: This article investigates the existence and uniqueness of solutions for fuzzy differential equations (FDEs), extending classical ordinary differential equation theory to fuzzy environments. We establish a rigorous mathematical framework including definitions, axioms, and properties of fuzzy numbers, derivatives, and integrals. Using the Banach Fixed-Point Theorem and Picard operator, we prove that under continuity and Lipschitz conditions, every fuzzy initial value problem admits a unique solution. Analytical solutions for a linear FDE demonstrate the theorem's applicability. Finally, we outline open problems and future research directions for fuzzy fractional systems, stochastic models, and advanced numerical methods.

Keywords: Fuzzy Differential Equations (FDEs), Existence and Uniqueness Theorem, Hukuhara Derivative, Fuzzy Metric Space, Picard Operator, Banach Fixed-Point Theorem, Fuzzy Initial Value Problem (FIVP), Lipschitz Continuity, Fuzzy Set Theory, Uncertainty Modeling

1. Introduction

The theory of Fuzzy Differential Equations (FDEs) emerged as an essential extension of classical differential equations to capture uncertainty and vagueness inherent in real-world dynamical systems. Introduced by Kaleva (1987) and Seikkala (1987), this field utilizes fuzzy set theory, pioneered by Zadeh (1965), to define derivatives and integrals of fuzzy-valued functions, thereby providing a robust framework for modeling processes where precise data is unavailable or imprecise [1,2].

The theory has also seen important contributions from Indian researchers in recent years: several works established existence and uniqueness results for fuzzy initial/boundary-value problems under various differentiability concepts (including granular/gH differentiability) and for higher-order and multi-point problems, broadening the analytical toolkit used to treat FDEs in engineering and applied contexts [3,4,5,6].

Definition 1.1 (Fuzzy Set): A fuzzy set $\tilde{A}$ in a universe X is characterized by a membership function

$$\mu_{\tilde{A}} : X \rightarrow [0,1],$$

where $\mu_{\tilde{A}}(x)$ represents the degree of membership of $x \in X$ in $\tilde{A}$.

Definition 1.2 (Fuzzy Number): A fuzzy number $\tilde{x}$ is a normalized, convex, and upper semi-continuous fuzzy set on $\mathbb{R}$, whose α -cuts

$$[\tilde{x}]_{\alpha} = \{x \in \mathbb{R} \mid \mu_{\tilde{x}}(x) \geq \alpha\}, \quad \alpha \in (0,1],$$

are compact intervals [1].

Definition 1.3 (Fuzzy-Valued Function): A mapping $x : [a, b] \rightarrow E^d$, where E^d denotes the space of all compact convex fuzzy subsets of $\mathbb{R}^d$, is called a fuzzy-valued function if for every $t \in [a, b]$, $x(t)$ is a fuzzy number and the mapping is continuous in the metric

$$C \in E^d$$

where d_H is the Hausdorff distance [2].

Axiom 1.1 (Hukuhara Difference): For fuzzy numbers $A, B \in E^d$, if there exists a fuzzy number $C \in E^d$ such that

$$A = B \oplus C,$$

then C is called the **Hukuhara difference** of A and B , denoted by $A \ominus_H B = C$. This axiom is foundational in defining fuzzy derivatives [7].

Definition 1.4 (Fuzzy Derivative): A fuzzy-valued function $x(t)$ is said to be **Hukuhara differentiable** at t_0 if the limit

$$x'(t_0) = \lim_{h \rightarrow 0} \frac{x(t_0 + h) \ominus_H x(t_0)}{h}$$

exists in E^d [8].

Proposition 1.1 (Linearity of Fuzzy Derivative): If $x_1(t)x$ and $x_2(t)x$ are Hukuhara differentiable fuzzy-valued functions and $a, b \in \mathbb{R}$, then

$$(ax_1 + bx_2)' = ax_1' + bx_2'.$$

Theorem 1.1 (Existence and Uniqueness): Consider the initial value problem

$$x'(t) = f(t, x(t)), \quad t \in [t_0, t_0 + h], \quad x(t_0) = x_0 \in E^d.$$

If f is continuous and satisfies a Lipschitz condition

$$D(f(t, x), f(t, y)) \leq L D(x, y)$$

then there exists a unique fuzzy-valued solution $x(t)$ on $[t_0, t_0 + h]$. [9]

This introduction establishes the foundational concepts of fuzzy sets, fuzzy derivatives, and the axiomatic approach leading to the existence and uniqueness theorem, which will be rigorously proved in subsequent sections.

2. Preliminaries and Mathematical Framework

To rigorously analyze existence and uniqueness of solutions to fuzzy differential equations (FDEs), we first establish the essential mathematical framework. This includes the structure of fuzzy number spaces, associated metrics, properties of fuzzy-valued functions, and fundamental operators used in fixed-point analysis.

2.1 Space of Fuzzy Numbers: Let E^d denote the set of all fuzzy numbers on $\mathbb{R}^d$.

Definition 2.1 (Fuzzy Number): A fuzzy set $\tilde{x} \in E^d$ is called a fuzzy number if:

1. **Normality:** There exists $x_0 \in \mathbb{R}^d$ such that $\mu_{\tilde{x}}(x_0) = 1$.

2. **Convexity:** For all $x_1, x_2 \in \mathbb{R}^d$,

$$\mu_{\tilde{x}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{x}}(x_1), \mu_{\tilde{x}}(x_2)\}, \quad \lambda \in [0, 1].$$

3. **Upper Semi-Continuity:** The membership function $\mu_{\tilde{x}}$ is upper semi-continuous.

4. **Compact Support:** The closure of the support $(\tilde{x}) = \{x \in \mathbb{R}^d : \mu_{\tilde{x}}(x) > 0\}$ is compact.

Every fuzzy number $\tilde{x}$ has α -cuts:

$$\tilde{x}_\alpha^* = \{x \in \mathbb{R}^d : \mu_{\tilde{x}}(x) \geq \alpha\}, \quad \alpha \in (0, 1],$$

which are nonempty, compact, and convex subsets of $\mathbb{R}^d$. [1]

2.2 Metric Structure: Define the Hausdorff metric d_H for two compact sets $A, B \subset \mathbb{R}^d$;

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}.$$

Definition 2.2 (Fuzzy Metric): For $\tilde{x}, \tilde{y} \in E^d$, the fuzzy metric is

$$D(\tilde{x}, \tilde{y}) = \sup_{\alpha \in [0, 1]} d_H(\tilde{x}_\alpha^*, \tilde{y}_\alpha^*).$$

This metric turns (E^d, D) into a complete metric space. [2]

2.3 Fuzzy-Valued Functions

Definition 2.3 (Fuzzy-Valued Function): A mapping $x : [a, b] \rightarrow E^d$ is a fuzzy-valued function if for every $\alpha \in [0, 1]$, the α -cut mapping

$$x_\alpha(t) = [x(t)]_\alpha = [x_\alpha^-(t), x_\alpha^+(t)] \subset \mathbb{R}^d$$

is continuous in t .

Proposition 2.1 (Continuity Equivalence): A fuzzy-valued function $x(t)$ is continuous in the fuzzy metric D if and only if each of its endpoint functions $x_\alpha^-(t)$ and $x_\alpha^+(t)$ is continuous for all $\alpha \in [0, 1]$. [10]

2.4 Fuzzy Integral

Definition 2.4 (Fuzzy Integral): The integral of a fuzzy-valued function $f : [a, b] \rightarrow E^d$ is defined levelwise:

$$\left[\int_a^b f(t) dt \right]_\alpha = \left[\int_a^b f_\alpha^-(t) dt, \int_a^b f_\alpha^+(t) dt \right],$$

for each $\alpha \in [0, 1]$. This integral exists whenever both endpoints are Lebesgue integrable. [11]

2.5 Hukuhara Derivative: To handle derivatives of fuzzy functions, we use the Hukuhara difference.

Definition 2.5 (Hukuhara Derivative): A fuzzy function $x : [a, b] \rightarrow E^d$ is Hukuhara differentiable at $t_0 \in [a, b]$ if there exists a fuzzy number $x'(t_0)$ such that

$$x'(t_0) = \lim_{h \rightarrow 0} \frac{x(t_0 + h) \ominus_H x(t_0)}{h},$$

where $\ominus_H$ denotes the Hukuhara difference. [9]

2.6 Picard Operator: Define the **Picard operator** - for the fuzzy initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0 :$$

$$(-x)(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds.$$

Proposition 2.2 (Contraction Mapping): If f satisfies a Lipschitz condition

$$D(f(t, x), f(t, y)) \leq L D(x, y), \quad \forall x, y \in E^d,$$

then the operator - is a contraction on the Banach space

$$(C([t_0, t_0 + h], E^d), \square \cdot \square_\infty), \quad \square x \square_\infty = \sup_{t \in [t_0, t_0 + h]} D(x(t), 0).$$

Thus, by the Banach Fixed-Point Theorem, - has a unique fixed point corresponding to the unique solution of the FDE. [9]

This framework establishes the necessary mathematical tools - metric structure, differentiability, and the Picard operator - required for proving the Existence and Uniqueness Theorem, which will be formally stated and proved in the next section.

3. Main Existence and Uniqueness Theorem

In this section, we rigorously establish the existence and uniqueness of solutions for fuzzy differential equations (FDEs) using the Banach Fixed-Point Theorem. This result extends the classical Picard–Lindelöf theorem to fuzzy-valued functions.

3.1 Statement of the Initial Value Problem

Consider the fuzzy initial value problem (FIVP):

$$x'(t) = f(t, x(t)), \quad t \in [t_0, t_0 + h],$$

$$x(t_0) = x_0 \in E^d,$$

where E^d is the complete metric space of fuzzy numbers (Section 2). The function

$$f : [t_0, t_0 + h] \times E^d \rightarrow E^d$$

is fuzzy-valued and levelwise continuous.

Assumptions

- **(H1) Continuity:** $f(t, x)$ is continuous in t and fuzzy-continuous in xxx on

$$J = [t_0, t_0 + h] \times B(x_0, b),$$

where $B(x_0, b)$ is the fuzzy ball of radius bbb .

- **(H2) Lipschitz Condition:** There exists a constant $L > 0$ such that

$$D(f(t, x), f(t, y)) \leq L D(x, y), \quad \forall x, y \in B(x_0, b), t \in [t_0, t_0 + a].$$

- **(H3) Boundedness:**

$$M = \sup\{D(f(t, x), 0) : (t, x) \in J\} < \infty.$$

Definition 3.1 (Solution of FDE): A fuzzy-valued function $x : [t_0, t_0 + h] \rightarrow E^d$ is a solution of the FIVP if:

1. $x(t)$ is Hukuhara differentiable on $[t_0, t_0 + h]$;
2. $x(t_0) = x_0$;
3. For all t ,

$$x'(t) = f(t, x(t)) \quad \text{in } E^d.$$

Theorem 3.1 (Existence and Uniqueness)

Under assumptions (H1)–(H3), there exists a unique fuzzy-valued function $x(t)$ solving the FIVP on

$$[t_0, t_0 + \delta], \quad \delta = \min\left\{a, \frac{b}{M}\right\}.$$

Proof. We outline the proof in steps:

Step 1: Transformation into Integral Equation

Define the Picard operator - as:

$$(-x)(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds.$$

A fixed point of $\mathcal{T}$ corresponds to a solution of the FDE.

Step 2: Function Space

Consider the Banach space

$$\mathcal{X} = \left\{ x : [t_0, t_0 + \delta] \rightarrow E^d \mid \|x\|_\infty = \sup_{t \in [t_0, t_0 + \delta]} D(x(t), 0) < \infty \right\}.$$

This space is complete with respect to the norm $\|x\|_\infty$ (Proposition 2.2).

Step 3: Contraction Mapping

For any $x, y \in \mathcal{X}$,

$$\begin{aligned} D((\mathcal{T}x)(t), (\mathcal{T}y)(t)) &= D\left(\int_{t_0}^t f(s, x(s))ds, \int_{t_0}^t f(s, y(s))ds\right) \\ &\leq \int_{t_0}^t D(f(s, x(s)), f(s, y(s)))ds \\ &\leq L \int_{t_0}^t D(x(s), y(s))ds \leq L\delta \|x - y\|_\infty. \end{aligned}$$

Choose δ small enough such that $q = L\delta < 1$, making $\mathcal{T}$ a contraction.

Step 4: Application of Banach Fixed-Point Theorem

Since $\mathcal{T}$ is a contraction on the complete space $\mathcal{X}$, by the Banach Fixed-Point Theorem, there exists a unique fixed point $x^* \in \mathcal{X}$ such that:

$$\mathcal{T}(x^*) = x^*,$$

which implies

$$x^*(t) = x_0 + \int_{t_0}^t f(s, x^*(s))ds.$$

Differentiating both sides (in the Hukuhara sense),

$$x^{*'}(t) = f(t, x^*(t)).$$

Thus, $x^*(t)$ is the unique solution.

Corollary 3.1 (Continuous Dependence): If the initial condition x_0 is perturbed by Δx_0 , the solution $x(t)$ changes continuously with respect to x_0 , i.e.,

$$D(x_1(t), x_2(t)) \leq e^{L(t-t_0)} D(x_0^1, x_0^2),$$

where $x_1(t)$ and $x_2(t)$ correspond to different initial conditions x_0^1, x_0^2 . [12]

Corollary 3.2 (Global Existence)

If f is globally Lipschitz on $[t_0, \infty) \times E^d$, then the solution exists and is unique on the entire interval $[t_0, \infty)$. [9]

Remark 3.1

- The theorem extends directly to fuzzy systems of equations by considering E^d for $d > 1$.
- The result also generalizes to fractional fuzzy differential equations with Caputo or Riemann–Liouville derivatives, provided similar Lipschitz conditions hold. [8]

4. Applications and Examples

The existence and uniqueness theorem for fuzzy differential equations (FDEs) has significant implications in applied mathematics, control theory, and systems modeling where uncertainty cannot be ignored. This section presents a practical example demonstrating how the theorem ensures a unique solution to a simple fuzzy initial value problem (FIVP).

4.1 Real-World Motivation

Uncertainty in dynamical systems arises naturally in various fields:

- Engineering:** Modeling heat transfer with imprecise thermal conductivity.
- Economics:** Forecasting price dynamics with vague or incomplete data.
- Biology:** Population growth models with uncertain reproduction rates.

Classical ODEs fail to incorporate such imprecision, but FDEs provide a rigorous alternative.

4.2 Example: Linear Fuzzy Differential Equation

Consider the FIVP:

$$x'(t) = a \otimes x(t), \quad t \geq 0, \quad x(0) = x_0,$$

where

- $a \in \mathbb{R}$ is a crisp constant,
- $x_0 \in E^1$ is a fuzzy number,

- $\otimes$ denotes scalar multiplication of a crisp number with a fuzzy number.

Step 1: Verifying the Hypotheses

Define

$$f(t, x) = a \otimes x.$$

- **Continuity:** The function is linear and hence continuous in both t and x .
- **Lipschitz condition:** For any $x, y \in E^1$:

$$D(f(t, x), f(t, y)) = a \otimes D(x, y),$$

so $L = |a|$.

Thus, assumptions (H1)–(H2) of Theorem 3.1 are satisfied, guaranteeing existence and uniqueness.

Step 2: Integral Representation

Using the Picard operator, the solution satisfies:

$$x(t) = x_0 + \int_0^t a \otimes x(s) ds.$$

Step 3: Solving Levelwise

Let $[x(t)]_\alpha = [x_\alpha^-(t), x_\alpha^+(t)]$ be the α -cut of $x(t)$.

From the derivative definition:

$$\frac{d}{dt} x_\alpha^-(t) = ax_\alpha^-(t), \quad \frac{d}{dt} x_\alpha^+(t) = ax_\alpha^+(t),$$

with initial conditions:

$$x_\alpha^-(0) = x_0^-(\alpha), \quad x_\alpha^+(0) = x_0^+(\alpha).$$

Step 4: Solution

The system is a pair of classical ODEs with solutions:

$$x_\alpha^-(t) = x_\alpha^-(\alpha)e^{at}, \quad x_\alpha^+(t) = x_\alpha^+(\alpha)e^{at}.$$

Thus,
$$[x(t)]_\alpha = [x_0^-(\alpha)e^{at}, x_0^+(\alpha)e^{at}], \quad \alpha \in [0, 1].$$

This defines the unique fuzzy solution:

$$x(t) = e^{at} \otimes x_0.$$

4.3 Interpretation

- If $a > 0$, the fuzzy solution expands over time, reflecting uncertainty growth in the system.
- If $a < 0$, the solution shrinks, and uncertainty diminishes.
- When $a = 0$, $x(t)$ remains constant and equal to x_0 .

4.4 Numerical Illustration

Suppose:

- $a = 1$,
- $x_0 = (1, 2, 3)$ (a triangular fuzzy number).

The solution is:

$$[x(t)]_\alpha = [1 + (2 - 1)\alpha, 3 - (3 - 2)\alpha] e^t,$$

showing how each α -cut evolves exponentially while preserving the fuzzy shape.

Corollary 4.1 (Nonlinear Example)

For a nonlinear FDE:

$$x'(t) = a \otimes x(t) \otimes b \otimes t, \quad x(0) = x_0,$$

existence and uniqueness follow from Theorem 3.1, and the solution can be obtained via variation of parameters, extending the classical method to fuzzy settings. [9, 12, 13]

4.5 Applications

- **Control Systems:** Designing controllers with uncertain sensor data.
- **Epidemiology:** Modeling infection rates with fuzzy parameters.
- **Economics:** Solving dynamic investment models with imprecise cost functions.

These examples underline the practical utility of existence-uniqueness theorems: they ensure that even under uncertainty, system evolution is well-defined and predictable.

5. Conclusion

In this article, we rigorously established the existence and uniqueness theorem for fuzzy differential equations (FDEs), extending classical differential equation theory to fuzzy-valued functions. We developed a comprehensive mathematical framework, introducing fuzzy numbers, fuzzy metrics, Hukuhara derivatives, and the Picard operator within a complete Banach space. Using the Banach Fixed-Point Theorem, we proved that under continuity and Lipschitz conditions, every fuzzy initial value problem has a unique solution. A linear FDE was analytically solved to demonstrate uncertainty propagation, ensuring that models in engineering, economics, and biology with fuzzy parameters remain mathematically sound, stable, and predictable under imprecise conditions.

References

- [1] Puri, M.L. & Ralescu, D.A. (1983). *Differentials of fuzzy functions*. Journal of Mathematical Analysis and Applications, 91(2), 552–558.
- [2] Goetschel, R. & Voxman, W. (1986). *Elementary fuzzy calculus*. Fuzzy Sets and Systems, 18(1), 31–43.
- [3] Soma, N., Grande, S. K., Agarwal, R. P., & Murty, M. S. N. (2023). Existence and uniqueness of solutions for fuzzy initial value problems under granular differentiability. Journal of Mathematics and Computer Science, 31(2), 197–213.
- [4] Soma, N., Grande, G. S., Agarwal, R. P., Wang, C., & Murty, M. S. N. (2023). Existence and uniqueness of solutions for fuzzy boundary value problems under granular differentiability. Fuzzy Information and Engineering, 15(3), 291–312. DOI:10.26599/FIE.2023.9270021.
- [5] Murty, K. N., & Sailaja, P. (2023). Existence and uniqueness of solutions to three-point boundary value problems associated with third-order non-linear fuzzy differential equations. International Journal of Engineering and Computer Science, 12(02), Article 25648-25653. DOI:10.18535/ijecs/v12i02.4719.
- [6] Leelavathi, R. (2019). Existence and uniqueness of solutions for fuzzy nabla dynamic equations on time scales under Hukuhara differentiability. Advances in Continuous and Discrete Models / SpringerOpen
- [7] Hukuhara, M. (1967). *Integration des applications mesurables dont la valeur est un compact convexe*. Funkcialaj Ekvacioj, 10(2), 205–223.
- [8] Bede, B. & Gal, S.G. (2005). *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*. Fuzzy Sets and Systems, 151(3), 581–599.
- [9] Lakshmikantham, V. & Mohapatra, R.N. (2003). *Theory of fuzzy differential equations and inclusions*. CRC Press.
- [10] Kaleva, O. (1987). *Fuzzy differential equations*. Fuzzy Sets and Systems, 24(3), 301–317.
- [11] Bede, B., & Stefanini, L. (2013). *Generalized differentiability of fuzzy-valued functions*. Fuzzy Sets and Systems, 230, 119–141.
- [12] Wu, H.C. (2004). *Fuzzy differential equations in metric spaces*. Fuzzy Sets and Systems, 114(3), 395–402.
- [13] Abbasbandy, S.; Allahviranloo, T. & Ahmady, M. (2009). *Numerical solutions of fuzzy differential equations by Taylor method*. Applied Mathematics and Computation, 166(2), 438–449.
