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Introduction to Certified Domination Polynomial of Graphs

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Abstract: Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a simple graph of order n. The certified domination polynomial of G is a polynomial that is defined as follows: it is based on the minimum number of vertices needed in a certified dominating set, which is called the certified domination number and the number of certified dominating set of a specific size. In this article, we obtain some properties of the certified domination polynomial and its coefficients.

***Index terms* - certified dominating set, certified domination number, certified domination polynomial**

1. INTRODUCTION

Let $G = (V, E)$ be a simple, finite, undirected connected graph without loops and multiple edges. G's order and size are shown by numbers n and m respectively. A vertex in a graph G has an open neighbouthood defined as the set $N_G(v) = \{u \in V(G) \,/\, uv \in E(G)\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$.

The concept of dominating sets and the domination number in Graph Theory were first introduced by Oystein Ore and Claude Berge in 1960s. The concept of domination polynomial in Graph Theory was introduced by Saied Alikhani and Yee-hock Peng in 2009[10]. The concept of certified domination in graphs was introduced by Dettlaf et al.,2020[5]. They also further studied the concept in their subsequent work including its applications in real life situations.

While extending the concept of domination polynomial and certified domination, we came across many interesting relations among the certified domination polynomial of different graphs. In this article, we study some of the properties of the coefficients of the certified domination polynomial and certified domination polynomial. We need some preliminaries to study the properties of certified domination polynomial of graphs.

Definition 1.1: [3] A set $S \subseteq V$ is a **dominating set** of G, if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of dominating set. A dominating set with cardinality $\gamma(G)$ is called a γ – set.

Definition 1.2: [2] A dominating set S is a **certified dominating set** of G if S has either zero or at least two neighbours in $V - S$. The certified domination number $\gamma_{cer}(G)$ of G is the minimum cardinality of certified dominating set. A certified dominating set with cardinality $\gamma_{cer}(G)$ is called a γ_{cer} – set.

Definition 1.3: A polynomial is said to be **integer monic** if all its coefficient are integers and the coefficient of the highest power (the leading coefficient) is one.

Definition 1.4 : A polynomial is said to be **primitive** is the greatest common divisor of its coefficient is one.

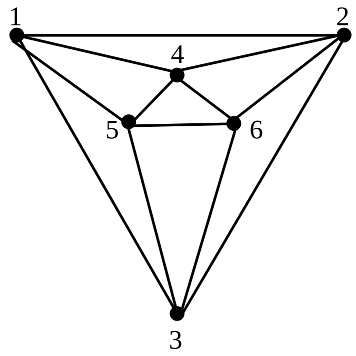
Definition 1.5: The **helm graph** H_n is a graph with $2n + 1$ vertices obtained from an $n -$ wheel graph by adjoining a pendant edge at each node of the cycle.

2. INTRODUCTION TO CERTIFIED DOMIANTION POLYNOMIAL

In this section, we state the definition of certified domination polynomial and its properties.

Definition 2.1: Let G be a simple connected graph. Let $D_{cer}(G, i)$ be a family of all certified dominating sets of G with cardinality i and let $d_{cer}(G, i) = |D_{cer}(G, i)|$. Then the **certified domination polynomial** $D_{cer}(G, x)$ is defined as $D_{cer}(G, x) = \sum_{i=\gamma_{cer}(G)}^{|V(G)|} d_{cer}(G, i)\, x^i$, where $\gamma_{cer}(G)$ is the certified domination number of G.

Example 2.2 : Consider the octahedral graph,



If $i = 1$ and 5, there are no certified dominating sets. If $i = 2$, then $(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6)$ and $(5,6)$ are the certified dominating sets. If $i = 3$, then $(1,2,3), (1,2,4), (1,2,5), (1,2,6), (1,3,4), (1,3,5), (1,3,6), (1,4,5), (1,4,6), (1,5,6), (2,3,4), (2,3,5), (2,3,6), (2,4,5), (2,4,6), (2,5,6), (3,4,5), (3,4,6), (3,5,6)$ and $(4,5,6)$ are the certified dominating sets. If $i = 4$ then $(1,2,5,6), (1,3,4,6)$ and $(2,3,4,5)$ are the certified dominating sets. If $i = 6$, then $(1,2,3,4,5,6)$ is the certified dominating set. Hence the certified domination polynomial of the octahedral graph is $15x^2 + 20x^3 + 3x^4 + x^6$

Theorem 2.3:

Let G be a simple connected graph with n vertices. Then

(i) The coefficient of x^n in the certified domination polynomial is one.

(ii) The certified dominating set of G with cardinality i is empty if and only if $i < \gamma_{cer}(G)$ or $i = n - 1$ or $i > n$.

(iii) The certified domination polynomial has no constant and $(n-1)^{th}$ term.

(iv) zero is the root of certified domination polynomial with multiplicity $\gamma_{cer}(G)$

Proof:

Let $G = (V, E)$ be a simple, connected graph with n vertices.

(i) There is only one possibility to choose all the vertices. Therefore the coefficient of x^n in the certified domination polynomial is one.

(ii) Clearly the certified dominating set of G with cardinality i is empty if and only if $i < \gamma_{cer}(G)$ or $i > n$. If we select a set of vertices with cardinality $n - 1$, then at least only one vertex in that set has one neighbour outside the set, which contradicts the definition of certified dominating set. Hence also for the cardinality $n - 1$, there are no certified dominating sets.

(iii) Since $\gamma_{cer}(G) \geq 1$, there are no constant term in the certified domination polynomial. Also by (ii), the certified domination polynomial has no $(n-1)^{th}$ term.

(iv) Since the certified domination polynomial has no constant term, $D_{cer}(G, x) = 0$ when $x = 0$. Also since the least power of $D_{cer}(G, x)$ is $\gamma_{cer}(G)$, zero is the root of certified domination polynomial with multiplicity $\gamma_{cer}(G)$

Theorem 2.4: Let G be a simple connected graph n vertices. Then

(i) The degree of the certified domination polynomial of G is the order of graph G.

(ii) If H is a sub graph of G, then the degree of the certified domination polynomial of H is less than or equal to the degree of the certified domination polynomial of G.

Proof:

(i) $\gamma_{cer}(G)$ is less than or equal to the order of G. Also there is only one certified dominating G with cardinality n and certified dominating set whose cardinality greater than n are always empty. Therefore maximum cardinality of the certified dominating set of G is n, which is the order of G. Therefore the degree of the certified domination polynomial of G is the order of graph G

(ii) Let H be a sub graph of G. Then the order of H is less than or equal to the order of G. Hence by (i), degree of the certified domination polynomial of H is less than or equal to the degree of the certified domination polynomial of G.

Theorem 2.5 :

Let G be a simple connected graph. Then the certified domination polynomial of G is both integer monic and primitive.

Proof:

Theorem 2.4(i) gives the degree of the certified domination polynomial of G is the order of G. Also all the coefficient of certified domination polynomial of G is a positive integer and the coefficient of the highest power is one. Hence certified domination polynomial is integer monic. Since the coefficient of the highest power is one, the greatest common divisor of all the coefficient of certified domination polynomial is one. Hence the certified domination polynomial of G is primitive.

Theorem 2.6:

Let G_1 and G_2 be two isomorphic graphs. Then the certified domination polynomial of G_1 and G_2 are same.

Proof:

Let G_1 and G_2 be two isomorphic graphs. Let $V(G_1) = \{v_1, v_2, \dots, v_n\}$ and $V(G_2) = \{f(v_1), f(v_2), \dots, f(v)_n\}$. Let $f: G_1 \to G_2$ be a isomorphism. Then the order, size and degree of the graphs G_1 and G_2 are equal. For any two vertices v_i and v_j in G_1, there is an edge between v_i and v_j if and only if there is an edge between $f(v_i)$ and $f(v_j)$. Hence the certified domination polynomial of G_1 and G_2 are equal.

Theorem 2.7 :

If a graph consists of m components $G_1, G_2, \dots, G_n$ then $D_{cer}(G^m, x)$ is equal to $D_{cer}(G_1, x) D_{cer}(G_2, x) \dots D_{cer}(G_m, x)$

Proof:

We prove this theorem by induction on n. When $m = 2$, for $k \geq \gamma_{cer}(G)$, a certified dominating set of k vertices in G arises by choosing a certified dominating set of j vertices in G_1 for some $j \in \gamma_{cer}(G_1), \gamma_{cer}(G_1) + 1, \dots., |V(G_1)|$ and a certified dominating set of $k - j$ vertices in G_2. The number of doing this over all $j = \gamma_{cer}(G_1), \gamma_{cer}(G_1) + 1, \dots., |V(G_1)|$ is exactly the coefficient of x^k in $D_{cer}(G_1, x) D_{cer}(G_2, x)$. Hence $D_{cer}(G^2, x)$ and $D_{cer}(G_1, x) D_{cer}(G_2, x)$ have the same coefficient, so they are identical polynomials.

Assume that this result is true for all the graphs having less than or equal to $m - 1$ components. That is $D_{cer}(G^{m-1}, x) = D_{cer}(G_1, x) D_{cer}(G_2, x) \dots D_{cer}(G_{m-1}, x)$. For n,

$D_{cer}(G_1, x) D_{cer}(G_2, x) \dots D_{cer}(G_m, x) = D_{cer}(G^{m-1}, x) D_{cer}(G_m, x) = D_{cer}(G^m, x)$ (by induction hypothesis)

Corollary 2.8:

Let $\overline{K_n}$ be the null graph with n vertices. Then $D_{cer}(\overline{K_n}, x) = x^n$

Proof :

Since $D_{cer}(\overline{K_n}, x) = x$, by the above theorem $D_{cer}(\overline{K_n}, x) = x^n$

Theorem 2.9 :

The certified domination polynomial of helm graph H_n is $x^n(1 + x^{n+1})$ for all $n \geq 3$.

Proof:

Let H_n be the helm graph with $2n + 1$ vertices. Let the vertex set of H_n be $\{v_j / 0 \leq j \leq 2n\}$, where v_0 is the center vertex, v_j $(0 \leq j \leq n)$ represent the vertices on the cycle and v_j $(n + 1 \leq j \leq 2n)$ represent the pendant vertices. Clearly $\{v_j / 1 \leq j \leq n\}$ is the minimum certified dominating set. Hence $\gamma_{cer}(H_n) = n$. If $i = n$, the minimum certified dominating set is the only certified dominating set of this size and if $i = 2n + 1$, $\{v_j / 0 \leq j \leq 2n\}$ is the certified dominating set. Also for the other values of i, there are no certified dominating sets. Therefore $D_{cer}(H_n, x) = x^n + x^{2n+1} = x^n(1 + x^{n+1})$.

Lemma 2.10:

(i) The sum of the coefficients of certified domination polynomial of H_n is two for all $n \geq 3$.

(ii) The roots of the certified domination polynomial of H_n is $0(n\, times)$ or $e^{i\frac{(2k+1)}{n+1}}$ for all $n \geq 3$.

Proof:

(i) is obvious

(ii) $x^n(1 + x^{n+1}) = 0$

$\Rightarrow x^n = 0$ or $(1 + x^{n+1}) = 0$

Which gives $x = 0(n\ times)$

Now, $x^{n+1} = -1$

$\Rightarrow x^{n+1} = 1(cos\pi + isin\pi)$

$\Rightarrow x = (cos\pi + isin\pi)^{\frac{1}{n+1}}$

$\Rightarrow x = cos\left(\frac{(2k+1)\pi}{n+1}\right) + isin\left(\frac{(2k+1)\pi}{n+1}\right)$, where $k = 0,1,2,\dots,n$

Using the Euler's formula $e^{i\theta} = cos\theta + isin\theta$, we can write the roots in the exponential form as $e^{i\frac{(2k+1)}{n+1}}$

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