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REACHABILITY ENERGY AND INDEX IN THE CONTEXT OF REGULAR GRAPH

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Abstract

In this paper, we introduce some closed neighbourhood degree-based reachability matrices of graph and obtain its Energy and Estrada index. Additionally, we find some closed neighbourhood degree-based reachability energies for regular graph. Also, we establish the bounds for this energy and index.

Keywords: Regular graph, Reachability matrix, Spectrum of a graph, Energy, Estrada Index.

1. INTRODUCTION

The **Energy** of a simple graph was introduced by Ivan Gutman in 1978 [14,15,18]. The Energy of a graph G, denoted by $E(G)$, is defined to be the sum of the absolute value of the eigenvalues of its adjacency matrix (i.e) $E(G)=\sum_{i=1}^{p}|\lambda_i|$. Various other energy measures based on different matrices have been discussed [5,17].

De la Pe˜na et.al., introduced the **Estrada index** of a graph in 2007[6]. Estrada index of the graph G is defined by $EE(G)=\sum_{i=1}^{p}e^{\lambda_i}$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_p$ are the eigenvalues of the adjacency matrix $A(G)$ of G.Denoting by $M_k(G)$ to the k-th moment of the graph G, we get $M_k(G) = \sum_{i=1}^{p}(\lambda_i)^k$and recalling the power series expansion of e^x, we have $EE = \sum_{k=0}^{\infty}\frac{M_k}{k!}$. It is well known that $M_k(G)$ is equal to the number of closed walks of length k of the graph G [12]. In fact, Estrada index of graphs has an important role in chemistry and physics and there exists a vast literature that studies this special index [7,8,9,10,11,13,16].

Bala et. al., introduced the Reachability degree sum matrix and its energy of a graph in 2024 [1], Reachability energy and Estrada index of a graph [2], Reachability degree sum Estrada index of a graph [3], Reachability average LH matrix and Inverse reachability degree sum matrix and their corresponding energy and Estrada index [4] in 2025.

Let $G(T,W)$ be a simple connected graph with p vertices and q edges. The **Reachability matrix [5]** $M_1(G) = (b_{ij})$ of a graph G is the $p \times p$ matrix with

$$M_1(G) = (b_{ij}) = \begin{cases} 1, & t_j \text{ is reachable from } t_i \\ 0, & \text{otherwise} \end{cases}.$$

Reachability degree sum matrix [1]:

$$M_2(G) = (b_{ij}) = \begin{cases} r_{ij} + d_i + d_j, & t_j \text{ is reachable from } t_i \\ 0, & \text{otherwise} \end{cases}$$

where $d_i = degree\ of\ the\ vertex\ t_i$ and $r_{ij} = \begin{cases} 1, & i \neq j \\ 0, & i = j \end{cases}$.

Reachability Average LH matrix [4]:

$$M_3(G) = (b_{ij}) = \begin{cases} \frac{1}{2}\left(lcm(d_i, d_j) + hcf(d_i, d_j)\right), & t_j \text{ is reachable from } t_i \\ 0 \quad , & otherwise \end{cases}.$$

where $d_i = degree\ of\ the\ vertex\ t_i$.

Inverse Reachability degree sum matrix [4]:

$$M_4(G) = (b_{ij}) = \begin{cases} \frac{1}{r_{ij}+d_i+d_j}, & t_j \text{ is reachable from } t_i \\ 0 \quad , & otherwise \end{cases}.$$

where $d_i = degree\ of\ the\ vertex\ t_i$ and $r_{ij} = \begin{cases} 1, i \neq j \\ 0, i = j \end{cases}$.

Energy and Estrada index of a graph based on the matrices discussed above are as follows,

$$For\ 1 \leq a \leq 4, \quad E_a(G) = \sum_{i=1}^{p} \left|\mu_i{}^{(a)}\right| \quad and \quad EE_a(G) = \sum_{i=1}^{p} e^{\mu_i{}^{(a)}},$$

where $\mu_1{}^{(a)} \geq \mu_2{}^{(a)} \geq \cdots \geq \mu_p{}^{(a)}$ are the eigenvalues of $M_a(G)$

The open neighbourhood degree-based reachability energies and indices of a graph was discussed in 2025 [19].

2. PROPOSED CONCEPT:

In this section, we propose some closed neighbourhood degree-based reachability matrices and their corresponding Energy and Estrada index for a graph along with the notations Also, we present several lemmas based on these matrices, which form the foundation for the results discussed in the following sections.

Let G be a finite, simple, undirected, connected graph with p vertices. Let $M_a(G)$ denote the matrix of a graph G. Since the matrix $M_a(G)$ under consideration are real and symmetric, its **eigenvalues** are real numbers and is denoted by $\mu_1{}^{(a)}, \mu_2{}^{(a)}, \ldots, \mu_p{}^{(a)}$, we label them in non- increasing order $\mu_1{}^{(a)} \geq \mu_2{}^{(a)} \geq \cdots \geq \mu_p{}^{(a)}$. (i.e.,) For $1 \leq i \leq p$, $\mu_i{}^{(a)}$ be the eigenvalues for $M_a(G)$. Also, we denote upper triangular part of a matrix $M_a(G)$ as $R_a = \sum_{1 \leq i < j \leq p} \left(b_{ij}\right)^2$. Now we denote $E_a(G)\ and\ EE_a(G)$ as Energy and Estrada index of a graph based on a matrix $M_a(G)$ respectively.

In a graph $G = (T, W)$, the **neighbourhood** of a vertex $t \in T(G)$ is the set of all **vertices that are adjacent** to t. For a vertex $t \in T(G)$ in a graph $G = (T, W)$, the **Closed Neighbourhood** of t is the set of all vertices adjacent to t, including t. The **Closed Neighbourhood Degree** of t is defined as the sum of the degrees of all vertices in the closed neighbourhood of t. It is denoted by $N_c[t]$.

Closed Neighbourhood Reachability sum matrix:

$$M_{10}(G) = (b_{ij}) = \begin{cases} N_c[t_i] + N_c[t_j], & t_j \text{ is reachable from } t_i \\ 0 \quad , & otherwise \end{cases}$$

Closed Neighbourhood Reachability sum energy: $E_{10}(G) = \sum_{i=1}^{p} \left|\mu_i{}^{(10)}\right|$

Closed Neighbourhood Reachability sum Estrada index: $EE_{10}(G) = \sum_{i=1}^{p} e^{\mu_i{}^{(10)}}$, where $\mu_1{}^{(10)} \geq \mu_2{}^{(10)} \geq \cdots \geq \mu_p{}^{(10)}$ are the eigenvalues of $M_{10}(G)$.

Closed Neighbourhood Reachability degree sum matrix:

$$M_{11}(G) = (b_{ij}) = \begin{cases} r_{ij} + N_c[t_i] + N_c[t_j], & t_j \ is \ reachable \ from \ t_i \\ 0 & , \quad otherwise \end{cases}$$

where $r_{ij} = \begin{cases} 1, i \neq j \\ 0, i = j \end{cases}$.

Closed Neighbourhood Reachability degree sum energy: $E_{11}(G) = \sum_{i=1}^{p} |\mu_i^{(11)}|$

Closed Neighbourhood Reachability degree sum Estrada index: $EE_{11}(G) = \sum_{i=1}^{p} e^{\mu_i^{(11)}}$, where $\mu_1^{(11)} \geq \mu_2^{(11)} \geq \cdots \geq \mu_p^{(11)}$ are the eigenvalues of $M_{11}(G)$.

Inverse Closed Neighbourhood Reachability sum matrix:

$$M_{12}(G) = (b_{ij}) = \begin{cases} \frac{1}{N_c[t_i]+N_c[t_j]}, & t_j \ is \ reachable \ from \ t_i \\ 0 & , \quad otherwise \end{cases}$$

Inverse Closed Neighbourhood Reachability sum energy: $E_{12}(G) = \sum_{i=1}^{p} |\mu_i^{(12)}|$

Inverse Closed Neighbourhood Reachability sum Estrada index: $EE_{12}(G) = \sum_{i=1}^{p} e^{\mu_i^{(12)}}$, where $\mu_1^{(12)} \geq \mu_2^{(12)} \geq \cdots \geq \mu_p^{(12)}$ are the eigenvalues of $M_{12}(G)$.

Inverse Closed Neighbourhood Reachability degree sum matrix:

$$M_{13}(G) = (b_{ij}) = \begin{cases} \frac{1}{r_{ij}+N_c[t_i]+N_c[t_j]}, & t_j \ is \ reachable \ from \ t_i \\ 0 & , \quad otherwise \end{cases},$$

where $r_{ij} = \begin{cases} 1, i \neq j \\ 0, i = j \end{cases}$.

Inverse Closed Neighbourhood Reachability degree sum energy: $E_{13}(G) = \sum_{i=1}^{p} |\mu_i^{(13)}|$

Inverse Closed Neighbourhood Reachability degree sum Estrada index: $EE_{13}(G) = \sum_{i=1}^{p} e^{\mu_i^{(13)}}$, where $\mu_1^{(13)} \geq \mu_2^{(13)} \geq \cdots \geq \mu_p^{(13)}$ are the eigenvalues of $M_{13}(G)$.

Closed Neighbourhood Reachability average LH matrix:

$$M_{14}(G) = (b_{ij}) = \begin{cases} \frac{1}{2}\left(lcm(N_c[t_i], N_c[t_j]) + hcf(N_c[t_i], N_c[t_j])\right), & t_j \ is \ reachable \ from \ t_i \\ 0 & , \quad otherwise \end{cases}$$

Closed Neighbourhood Reachability average LH energy: $E_{14}(G) = \sum_{i=1}^{p} |\mu_i^{(14)}|$

Closed Neighbourhood Reachability average LH Estrada index: $EE_{14}(G) = \sum_{i=1}^{p} e^{\mu_i^{(14)}}$, where $\mu_1^{(14)} \geq \mu_2^{(14)} \geq \cdots \geq \mu_p^{(14)}$ are the eigenvalues of $M_{14}(G)$.

2.1 AUXILARY LEMMAS

In this subsection, we discuss some lemmas that are used to establish the bounds for largest eigenvalue for our proposed closed neighbourhood degree-based reachability matrices.

Lemma 2.1.1:

Let G be a connected graph of order p. For $10 \leq a \leq 14 \ and \ 1 \leq i \leq p$, $\mu_i^{(a)}$ be the eigenvalues for M_a then

$$tr\left(M_a(G)\right) = \sum_{i=1}^{p} \mu_i^{(a)} = 0$$

$$tr\left(M_a(G)^2\right) = \sum_{i=1}^{p} (\mu_i^{(a)})^2 = 2\,R_a, where\ R_a = \sum_{1\le i<j\le p} \left(b_{ij}\right)^2$$

Proof:

By usual notation, we have $\sum_{i=1}^{p} \mu_{\mathrm{i}}^{(a)}$ is equal to the trace of a matrix.

Now, $\sum_{i=1}^{p} \mu_{\mathrm{i}}^{(a)} = trace\left(M_a(G)\right) = \sum_{i=j=1}^{p} R_a = 0.$

Moreover, for $i = 1,2,\dots,p$, the $(i,i)th$ entry of $(M_a(G))^2$ is equal to $\sum_{j=1}^{p}\left(b_{ij}\right)\left(b_{ji}\right) = \sum_{i=1}^{p}(b_{ij})^2$

$$\sum_{i=1}^{p}(\mu_{\mathrm{i}}^{(a)})^2 = trace\,(M_a(G))^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}(b_{ij})^2 = 2\,R_a.$$

Hence the result.

Lemma 2.1.2:

Let G represents a connected graph with p vertices which satisfies the inequality

$$|Det\,(M_a(G))| \le (2\,R_a)^{\frac{p}{2}}, \text{for} 10 \le a \le 14.$$

Proof:

We know that, $|Det\,(M_a(G))| = \prod_{i=1}^{p}\left|(\mu_i^{(a)})\right|$

$$= \left|(\mu_1^{(a)})\right|\left|(\mu_2^{(a)})\right| \dots \left|(\mu_p^{(a)})\right| \le \left|(\mu_1^{(a)})\right|\left|(\mu_1^{(a)})\right| \dots \left|(\mu_1^{(a)})\right| \le \left|(\mu_1^{(a)})\right|^p$$

$$\le \left(\sqrt{2\sum_{1\le i<j\le p}\left(b_{ij}\right)^2}\right)^p \le (2\,R_a)^{\frac{p}{2}}$$

$$\therefore |Det\,(M_a(G))| \le (2\,R_a)^{\frac{p}{2}}.$$

Lemma 2.1.3:

If G is a connected graph with p vertices then $\mu_1^{(a)} \le \sqrt{\frac{2\,(p-1)R_a}{p}}$ for $10 \le a \le 14$.

Proof:

Using *Cauchy Schwartz inequality*, by setting $x_i = 1$ and $y_i = \mu_i^{(a)}$ for $i = 2,3,\dots,p$ then we get,

$$\left(\sum_{i=2}^{p}\left|\mu_i^{(a)}\right|\right)^2 \le (p-1)\left(\sum_{i=1}^{p}\mu_i^{(a)^2}\right)$$

$$\left(-\mu_1^{(a)}\right)^2 \le (p-1)\left(2\sum_{1\le i<j\le p}\left(b_{ij}\right)^2 - \mu_1^{(a)^2}\right)$$

$$\Rightarrow \mu_1^{(a)} \le \sqrt{\frac{2\,(p-1)R_a}{p}}.$$

Lemma 2.1.4:

Consider a connected graph G with p vertices. The largest eigenvalue $\mu_1^{(a)}$ of G satisfies the inequality $\left|\mu_1^{(a)}\right| \ge \ |Det\ \ (M_a(G))|^{\frac{1}{p}}\ for\ 10 \le a \le 14$.

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $\left|\mu_1^{(a)}\right|, \left|\mu_2^{(a)}\right|, \dots, \left|\mu_p^{(a)}\right|$, we get,

$$\frac{\left|\mu_1^{(a)} + \mu_2^{(a)} + \cdots + \mu_p^{(a)}\right|}{p} \ge \left|\mu_1^{(a)}\mu_2^{(a)} \dots\ \mu_p^{(a)}\right|^{\frac{1}{p}}$$

$$\Rightarrow \frac{\left|\mu_1^{(a)}\right| + \left|\mu_1^{(a)}\right| + \cdots + \left|\mu_1^{(a)}\right|}{p} \ge\ |Det\ M_a(G)|^{\frac{1}{p}}$$

$$\left|\mu_1^{(a)}\right| \ge\ |Det\ (M_a(G))|^{\frac{1}{p}}.$$

Lemma 2.1.5:

Let G be a connected graph with p vertices, the largest eigenvalue $\mu_1^{(a)}$ of G satisfies the inequality $\left|\mu_1^{(a)}\right| \ge \frac{|Det\ (M_a(G))|^{\frac{1}{p}}}{\sqrt{p}}\ for\ 10 \le a \le 14$.

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $\left|\mu_1^{(a)}\right|, \left|\mu_2^{(a)}\right|, \dots, \left|\mu_p^{(a)}\right|$, we get,

$$\frac{\left|\mu_1^{(a)} + \mu_2^{(a)} + \cdots + \mu_p^{(a)}\right|}{p} \ge \left|\mu_1^{(a)}\mu_2^{(a)} \dots\ \mu_p^{(a)}\right|^{\frac{1}{p}}$$

$$\frac{\left|\mu_1^{(a)}\right| + \left|\mu_1^{(a)}\right| + \cdots + \left|\mu_1^{(a)}\right|}{\sqrt{p}} \ge \frac{\left|\mu_1^{(a)}\right| + \left|\mu_1^{(a)}\right| + \cdots + \left|\mu_1^{(a)}\right|}{p} \ge \left|\mu_1^{(a)}\mu_2^{(a)} \dots\ \mu_p^{(a)}\right|^{\frac{1}{p}}$$

$$\Rightarrow \left|\mu_1^{(a)}\right| \ge \frac{|Det\ (M_a(G))|^{\frac{1}{p}}}{\sqrt{p}}.$$

3. CLOSED NEIGHBOURHOOD DEGREE BASED REACHABILITY ENERGY AND ITS BOUNDS

A **regular graph** is a graph in which all vertices have the same degree. If every vertex in a graph has degree k, then the graph is called a **k-regular** graph. In this section, we obtain some closed neighbourhood degree-based reachability energy of regular graph and calculate its bounds.

3.1. CLOSED NEIGHBOURHOOD DEGREE BASED REACHABILITY ENERGY OF REGULAR GRAPH

In this subsection, we find some closed neighbourhood degree-based reachability energies $E_a(G), 10 \le a \le 14$ for a regular graph.

Theorem 3.1.1:

Let G be a k- regular graph of order p then

(i) $E_{10}(G) = 4(k^2+k)(p-1)$

(ii) $E_{11}(G) = 2\,(2(k^2+k)+1)(p-1)$

(iii) $E_{12}(G) = \frac{p-1}{k^2+k}$

(iv) $E_{13}(G) = \frac{2(p-1)}{2(k^2+k)+1}$

(v) $E_{14}(G) = 2(k^2+k)(p-1)$

Proof:

Consider k- Regular graph G with p vertices.

(i) The closed neighbourhood reachability sum matrix $\mathrm{M}_{10}(G)$ is

$$\mathrm{M}_{10}(G) = \begin{pmatrix} 0 & 2(k^2+k) & 2(k^2+k) & \cdots & 2(k^2+k) & 2(k^2+k) \\ 2(k^2+k) & 0 & 2(k^2+k) & \cdots & 2(k^2+k) & 2(k^2+k) \\ 2(k^2+k) & 2(k^2+k) & 0 & \cdots & 2(k^2+k) & 2(k^2+k) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(k^2+k) & 2(k^2+k) & 2(k^2+k) & \cdots & 0 & 2(k^2+k) \\ 2(k^2+k) & 2(k^2+k) & 2(k^2+k) & \cdots & 2(k^2+k) & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_{10}(G)$ using the relation,

$$\phi(G,\mu) = \det(\mathrm{M}_{10}(G) - \mu I)\,, where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu) = \begin{vmatrix} -\mu & 2(k^2+k) & 2(k^2+k) & \cdots & 2(k^2+k) & 2(k^2+k) \\ 2(k^2+k) & -\mu & 2(k^2+k) & \cdots & 2(k^2+k) & 2(k^2+k) \\ 2(k^2+k) & 2(k^2+k) & -\mu & \cdots & 2(k^2+k) & 2(k^2+k) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(k^2+k) & 2(k^2+k) & 2(k^2+k) & \cdots & -\mu & 2(k^2+k) \\ 2(k^2+k) & 2(k^2+k) & 2(k^2+k) & \cdots & 2(k^2+k) & -\mu \end{vmatrix} = 0$$

Hence, the spectrum of $\mathrm{M}_{10}(G)$ is

$$\begin{pmatrix} -2(k^2+k) & 2(k^2+k)(p-1) \\ p-1 & 1 \end{pmatrix}.$$

The closed neighbourhood reachability sum energy $\mathrm{E}_{10}(G)$ can be determined as follows:

$$\mathrm{E}_{10}(G) = \sum_{i=1}^{p} \left|\mu_{\mathrm{i}}{}^{(10)}\right|$$

$$= (|-2(k^2+k)| \times (p-1)) + (|2(k^2+k)(p-1)| \times 1)$$

$$\mathrm{E}_{10}(G) = 4(k^2+k)(p-1).$$

(ii) The closed neighbourhood reachability degree sum matrix $\mathrm{M}_{11}(G)$ is

$$\mathrm{M}_{11}(G) = \begin{pmatrix} 0 & 2(k^2+k)+1 & 2(k^2+k)+1 & \cdots & 2(k^2+k)+1 & 2(k^2+k)+1 \\ 2(k^2+k)+1 & 0 & 2(k^2+k)+1 & \cdots & 2(k^2+k)+1 & 2(k^2+k)+1 \\ 2(k^2+k)+1 & 2(k^2+k)+1 & 0 & \cdots & 2(k^2+k)+1 & 2(k^2+k)+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(k^2+k)+1 & 2(k^2+k)+1 & 2(k^2+k)+1 & \cdots & 0 & 2(k^2+k)+1 \\ 2(k^2+k)+1 & 2(k^2+k)+1 & 2(k^2+k)+1 & \cdots & 2(k^2+k)+1 & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_{11}(G)$ using the relation,

$$\phi(G,\mu)=\det(\mathrm{M}_{11}(G)-\mu I)\,, where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu)=\begin{vmatrix} -\mu & 2(k^2+k)+1 & 2(k^2+k)+1 & \cdots & 2(k^2+k)+1 & 2(k^2+k)+1 \\ 2(k^2+k)+1 & -\mu & 2(k^2+k)+1 & \cdots & 2(k^2+k)+1 & 2(k^2+k)+1 \\ 2(k^2+k)+1 & 2(k^2+k)+1 & -\mu & \cdots & 2(k^2+k)+1 & 2(k^2+k)+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(k^2+k)+1 & 2(k^2+k)+1 & 2(k^2+k)+1 & \cdots & -\mu & 2(k^2+k)+1 \\ 2(k^2+k)+1 & 2(k^2+k)+1 & 2(k^2+k)+1 & \cdots & 2(k^2+k)+1 & -\mu \end{vmatrix}=0$$

Hence, the spectrum of $\mathrm{M}_{11}(G)$ is

$$\begin{pmatrix} -(2(k^2+k)+1) & (2(k^2+k)+1)(p-1) \\ p-1 & 1 \end{pmatrix}.$$

The closed neighbourhood reachability degree sum energy $\mathrm{E}_{11}(G)$ can be determined as follows:

$$\mathrm{E}_{11}(G)=\sum_{i=1}^{p}\left|\mu_{\mathrm{i}}{}^{(11)}\right|$$

$$=(|-(2(k^2+k)+1)|\times(p-1))+(|(2(k^2+k)+1)(p-1)|\times 1)$$

$$\mathrm{E}_{11}(G)=2(2(k^2+k)+1)(p-1).$$

(iii) The inverse closed neighbourhood reachability sum matrix $\mathrm{M}_{12}(G\,)$ is

$$\mathrm{M}_{12}(G)=\begin{pmatrix} 0 & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \cdots & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} \\ \frac{1}{2(k^2+k)} & 0 & \frac{1}{2(k^2+k)} & \cdots & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} \\ \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & 0 & \cdots & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \cdots & 0 & \frac{1}{2(k^2+k)} \\ \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \cdots & \frac{1}{2(k^2+k)} & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_{12}(G)$ using the relation,

$$\phi(G,\mu)=\det(\mathrm{M}_{12}(G)-\mu I)\,, where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu)=\begin{vmatrix} -\mu & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \cdots & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} \\ \frac{1}{2(k^2+k)} & -\mu & \frac{1}{2(k^2+k)} & \cdots & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} \\ \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & -\mu & \cdots & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \cdots & -\mu & \frac{1}{2(k^2+k)} \\ \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \frac{1}{2(k^2+k)} & \cdots & \frac{1}{2(k^2+k)} & -\mu \end{vmatrix}=0$$

Hence, the spectrum of $\mathrm{M}_{12}(G)$ is

$$\begin{pmatrix} \frac{-1}{2(k^2+k)} & \frac{p-1}{2(k^2+k)} \\ p-1 & 1 \end{pmatrix}.$$

The inverse closed neighbourhood reachability sum energy $\mathrm{E}_{12}(G)$ can be determined as follows:

$$\mathrm{E}_{12}(G) = \sum_{i=1}^{p} \left|\mu_{\mathrm{i}}{}^{(12)}\right|$$

$$= \left(\left|\frac{-1}{2(k^2+k)}\right| \times (p-1)\right) + \left(\left|\frac{p-1}{2(k^2+k)}\right| \times 1\right)$$

$$\mathrm{E}_{12}(G) = \frac{p-1}{(k^2+k)}.$$

(iv) The inverse closed neighbourhood reachability degree sum matrix $\mathrm{M}_{13}(G)$ is

$$\mathrm{M}_{13}(G) = \begin{pmatrix} 0 & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \cdots & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} \\ \frac{1}{2(k^2+k)+1} & 0 & \frac{1}{2(k^2+k)+1} & \cdots & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} \\ \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & 0 & \cdots & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \cdots & 0 & \frac{1}{2(k^2+k)+1} \\ \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \cdots & \frac{1}{2(k^2+k)+1} & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_{13}(G)$ using the relation,

$$\phi(G,\mu) = \det(\mathrm{M}_{13}(G) - \mu I), where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu) = \begin{vmatrix} -\mu & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \cdots & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} \\ \frac{1}{2(k^2+k)+1} & -\mu & \frac{1}{2(k^2+k)+1} & \cdots & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} \\ \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & -\mu & \cdots & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \cdots & -\mu & \frac{1}{2(k^2+k)+1} \\ \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \frac{1}{2(k^2+k)+1} & \cdots & \frac{1}{2(k^2+k)+1} & -\mu \end{vmatrix} = 0$$

Hence, the spectrum of $\mathrm{M}_{13}(G)$ is

$$\begin{pmatrix} \frac{-1}{2(k^2+k)+1} & \frac{p-1}{2(k^2+k)+1} \\ p-1 & 1 \end{pmatrix}.$$

The inverse closed neighbourhood reachability degree sum energy $\mathrm{E}_{13}(G)$ can be determined as follows:

$$\mathrm{E}_{13}(G) = \sum_{i=1}^{p} \left|\mu_{\mathrm{i}}{}^{(13)}\right|$$

$$= \left(\left|\frac{-1}{2(k^2+k)+1}\right| \times (p-1)\right) + \left(\left|\frac{p-1}{2(k^2+k)+1}\right| \times 1\right)$$

$$\mathrm{E}_{13}(G) = \frac{2(p-1)}{2(k^2+k)+1}.$$

(v) The closed neighbourhood reachability average LH matrix $\mathrm{M}_{14}(G)$ is

$$\mathrm{M}_{14}(G)=\begin{pmatrix} 0 & k^2+k & k^2+k & \cdots & k^2+k & k^2+k \\ k^2+k & 0 & k^2+k & \cdots & k^2+k & k^2+k \\ k^2+k & k^2+k & 0 & \cdots & k^2+k & k^2+k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k^2+k & k^2+k & k^2+k & \cdots & 0 & k^2+k \\ k^2+k & k^2+k & k^2+k & \cdots & k^2+k & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_{14}(G)$ using the relation,

$$\phi(G,\mu)=\det(\mathrm{M}_{14}(G)-\mu I)\,, where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu)=\begin{vmatrix} -\mu & k^2+k & k^2+k & \cdots & k^2+k & k^2+k \\ k^2+k & -\mu & k^2+k & \cdots & k^2+k & k^2+k \\ k^2+k & k^2+k & -\mu & \cdots & k^2+k & k^2+k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k^2+k & k^2+k & k^2+k & \cdots & -\mu & k^2+k \\ k^2+k & k^2+k & k^2+k & \cdots & k^2+k & -\mu \end{vmatrix}=0$$

Hence, the spectrum of $\mathrm{M}_{14}(G)$ is

$$\begin{pmatrix} -(k^2+k) & (k^2+k)(p-1) \\ p-1 & 1 \end{pmatrix}.$$

The closed neighbourhood reachability average LH energy $\mathrm{E}_{14}(G)$ can be determined as follows:

$$\mathrm{E}_{14}(G)=\sum_{i=1}^{p}\left|\mu_{\mathrm{i}}{}^{(14)}\right|$$

$$=(|-(k^2+k)|\times(p-1))+(|(k^2+k)(p-1)|\times 1)$$

$$\mathrm{E}_{14}(G)=2(k^2+k)(p-1).$$

3.2 BOUNDS FOR CLOSED NEIGHBOURHOOD DEGREE BASED REACHABILITY ENERGY OF GRAPH

In this subsection, we obtain bounds for closed neighbourhood degree-based reachability energy of regular graph.

Theorem 3.2.1:

If G be a connected graph, then $\sqrt{2R_a}\le E_a(G)\le\sqrt{2pR_a}$ for $10\le a\le 14$.

Proof:

By *Cauchy-Schwartz inequality*,

$$\left(\sum_{i=1}^{p}x_iy_i\right)^2\le\left(\sum_{i=1}^{p}{x_i}^2\right)\left(\sum_{i=1}^{p}{y_i}^2\right)$$

Consider, $x_i=1$ and $y_i=\left|\mu_i{}^{(a)}\right|$, then

$$\left(\sum_{i=1}^{p}\left|\mu_i{}^{(a)}\right|\right)^2\le p\left(\sum_{i=1}^{p}(\mu_i{}^{(a)})^2\right)$$

$$E_a(G)^2\le 2pR_a$$

$$E_a(G) \leq \sqrt{2pR_a}.$$

which gives the required upper bound for $E_a(G)$.

Consider, $\left(E_a(G)\right)^2 = \left(\sum_{i=1}^{p}\left|\mu_i{}^{(a)}\right|\right)^2 \geq \sum_{i=1}^{p}\left|\mu_i{}^{(a)}\right|^2 = 2R_a$

$$E_a(G) \geq \sqrt{2R_a}.$$

which gives the required lower bound for $E_a(G)$.

$$\sqrt{2R_a} \leq E_a(G) \leq \sqrt{2pR_a}.$$

Hence the result.

Theorem 3.2.2:

Let G be a connected graph and let $|Det\ (M_a(G))|$ be the absolute value of the determinant of the $M_a(G)$ of a graph then for $10 \leq a \leq 14$,

$$\sqrt{2R_a + p|Det\ (M_a(G))|^{\frac{2}{p}}} \leq E_a(G) \leq \sqrt{2pR_a}.$$

Proof:

By the theorem 3.2.1, we have the upper bound for $E_a(G)$ as $E_a(G) \leq \sqrt{2pR_a}$.

Now, we obtain the lower bound for $E_a(G)$.

Consider,

$$\left(E_a(G)\right)^2 = \left(\sum_{i=1}^{p}\left|\mu_i{}^{(a)}\right|\right)^2 = \sum_{i=1}^{p}\left|\mu_i{}^{(a)}\right|^2 + 2\sum_{1\leq i<j\leq p}\left|\mu_i{}^{(a)}\right|\left|\mu_j{}^{(a)}\right|$$

$$= 2R_a + \sum_{i\neq j}\left|\mu_i{}^{(a)}\right|\left|\mu_j{}^{(a)}\right|$$

From *Arithmetic – Geometric mean inequality*, we have,

$$\frac{1}{p(p-1)}\sum_{i\neq j}\left|\mu_i{}^{(a)}\right|\left|\mu_j{}^{(a)}\right| \geq \left(\prod_{i\neq j}\left|\mu_i{}^{(a)}\right|\left|\mu_j{}^{(a)}\right|\right)^{\frac{1}{p(p-1)}}$$

$$= \left(\prod_{i=1}^{p}\left|\mu_i{}^{(a)}\right|^{2(p-1)}\right)^{\frac{1}{p(p-1)}} = |Det\ (M_a(G))|^{\frac{2}{p}}.$$

which implies that

$$\left(E_a(G)\right)^2 \geq 2R_a + p|Det\ (M_a(G))|^{\frac{2}{p}}$$

$$E_a(G) \geq \sqrt{2R_a + p|Det\ (M_a(G))|^{\frac{2}{p}}}\ .$$

which gives the required lower bound for $E_a(G)$.

$$\sqrt{2R_a + p|Det\ (M_a(G))|^{\frac{2}{p}}} \le E_a(G) \le \sqrt{2pR_a}.$$

Hence the result.

Theorem 3.2.3:

Let G be a connected graph with p vertices and $M_a(G)$ be a non-singular matrix then for $10 \le a \le 14$,

$$p|Det\ (M_a(G))|^{\frac{1}{p}} \le E_a(G) \le \frac{2pR_a}{|Det\ (M_a(G))|^{\frac{1}{p}}}.$$

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $|\mu_1{}^{(a)}|, |\mu_2{}^{(a)}|, \dots, |\mu_p{}^{(a)}|$, we get,

$$\frac{|\mu_1{}^{(a)} + \mu_2{}^{(a)} + \cdots + \mu_p{}^{(a)}|}{p} \ge |\mu_1{}^{(a)}\mu_2{}^{(a)} \dots \mu_p{}^{(a)}|^{\frac{1}{p}}$$

$$\sum_{i=1}^{p}|\mu_i{}^{(a)}| \ge p|Det\ (M_a(G))|^{\frac{1}{p}}$$

$$E_a(G) \ge p|Det\ (M_a(G))|^{\frac{1}{p}}$$

which gives a lower bound for $E_a(G)$.

By using lemma 2.1.4, we have $|\mu_1{}^{(a)}| \ge |Det\ (M_a(G))|^{\frac{1}{p}}$

$$|\mu_1{}^{(a)}|\sum_{i=1}^{p}|\mu_i{}^{(a)}| \ge |Det\ (M_a(G))|^{\frac{1}{p}} \sum_{i=1}^{p}|\mu_i{}^{(a)}|$$

$$\Rightarrow p|\mu_1{}^{(a)}|^2 \ge |Det\ (M_a(G))|^{\frac{1}{p}} \left(E_a(G)\right)$$

$$E_a(G) \le \frac{p|\mu_1{}^{(a)}|^2}{|Det\ (M_a(G))|^{\frac{1}{p}}}$$

Since $|\mu_1{}^{(a)}|^2 \le 2R_a$,

$$E_a(G) \le \frac{2pR_a}{|Det\ (M_a(G))|^{\frac{1}{p}}}$$

which gives an upper bound for $E_a(G)$.

$$p|Det\ (M_a(G))|^{\frac{1}{p}} \le E_a(G) \le \frac{2pR_a}{|Det\ (M_a(G))|^{\frac{1}{p}}}.$$

Hence the theorem.

4. CLOSED NEIGHBOURHOOD DEGREE BASED REACHABILITY ESTRADA INDEX AND ITS BOUNDS

In this section, we introduce and obtain some closed neighbourhood degree-based reachability Estrada index and its bounds. Additionally, we determine an upper bound for closed neighbourhood degree-based reachability Estrada index in terms of closed neighbourhood degree-based reachability energy of graph in the following sections.

4.1. CLOSED NEIGHBOURHOOD DEGREE BASED REACHABILITY ESTRADA INDEX OF GRAPH

In this subsection, we introduce some closed neighbourhood degree-based reachability Estrada index of a graph and derive bounds for the same.

Definition 4.1.1:

If G be a connected graph with p vertices, then the Estrada index $EE_a(G)$ based on a matrix $M_a(G)$ for $10 \leq a \leq 14$ is defined by

$$EE_a(G) = \sum_{i=1}^{p} e^{\mu_i^{(a)}}$$

where $\mu_1^{(a)} \geq \mu_2^{(a)} \geq \cdots \geq \mu_p^{(a)}$ are the eigenvalues of $M_a(G)$.

Denoting by $V_k(G)$ to the k-th moment of the graph G,

We get $V_k = \sum_{i=1}^{p} (\mu_i^{(a)})^k$, For $k = 0,1,2$

$$V_0 = \sum_{i=1}^{p} (\mu_i^{(a)})^0 = p; V_1 = \sum_{i=1}^{p} (\mu_i^{(a)})^1 = 0; V_2 = \sum_{i=1}^{p} (\mu_i^{(a)})^2 = 2 \sum_{1 \leq i < j \leq p} (b_{ij})^2 = 2R_a$$

Also, we have, $V_k = tr\ (M_a(G)^k)$. Then, $EE_a(G) = \sum_{k=0}^{\infty} \frac{V_k}{k!}$.

4.2. BOUNDS FOR CLOSED NEIGHBOURHOOD DEGREE BASED REACHABILITY ESTRADA INDEX OF GRAPH

In this subsection, we obtain the upper bound and lower bound for closed neighbourhood degree-based reachability Estrada index of graph.

Theorem 4.2.1:

Let G be a connected graph with diameter less than or equal to 2 then for $10 \leq a \leq 14$,

$$\sqrt{p^2 + 4\mathrm{R}_a} \leq EE_a(G) \leq p - 1 + e^{\sqrt{2\mathrm{R}_a}}.$$

Proof:

From the definition 4.1.1, $EE_a(G) = \sum_{i=1}^{p} e^{\mu_i^{(a)}}$

$$EE_a{}^2(G) = \left(\sum_{i=1}^{p} e^{\mu_i^{(a)}} \right)^2 = \sum_{i=1}^{p} e^{2\mu_i^{(a)}} + 2 \sum_{1 \leq i < j \leq p} e^{\mu_i^{(a)}} e^{\mu_j^{(a)}}$$

Consider the 2nd term of the above equation, by using *Arithmetic-Geometric Mean Inequality*, we have

$$2 \sum_{1 \leq i < j \leq p} e^{\mu_i^{(a)}} e^{\mu_j^{(a)}} \geq p(p-1) \left(\prod_{1 \leq i < j \leq p} e^{\mu_i^{(a)}} e^{\mu_j^{(a)}} \right)^{\frac{2}{p(p-1)}}$$

$$= p(p-1)\left(\left(\prod_{i=1}^{p} e^{\mu_i^{(a)}}\right)^{p-1}\right)^{\frac{2}{p(p-1)}}$$

$$= p(p-1)(e^{V_1})^{\frac{2}{p}} = p(p-1)$$

$$2\sum_{1\le i<j\le p} e^{\mu_i^{(a)}} e^{\mu_j^{(a)}} \ge p(p-1).$$

Consider,

$$\sum_{i=1}^{p} e^{2\mu_i^{(a)}} = \sum_{i=1}^{p}\sum_{k\ge 0}\frac{\left(2\mu_i^{(a)}\right)^k}{k!} = p + 4R_a + \sum_{i=1}^{p}\sum_{k\ge 3}\frac{\left(2\mu_i^{(a)}\right)^k}{k!}$$

Since we require lower bound as good as possible, it holds reasonable to replace $\sum_{k\ge 3}\frac{\left(2\mu_i^{(a)}\right)^k}{k!}$ by $4\frac{\left(\mu_i^{(a)}\right)^k}{k!}$. Further, we use a multiplier $t \in [0,4]$ instead of 4. We get,

$$\sum_{i=1}^{p} e^{2\mu_i^{(a)}} \ge p + 4R_a + t\sum_{i=1}^{p}\sum_{k\ge 3}\frac{\left(\mu_i^{(a)}\right)^k}{k!}$$

$$\sum_{i=1}^{p} e^{2\mu_i^{(a)}} \ge p + 4R_a - tp - tR_a + t\sum_{i=1}^{p}\sum_{k\ge 0}\frac{\left(\mu_i^{(a)}\right)^k}{k!}$$

$$\sum_{i=1}^{p} e^{2\mu_i^{(a)}} \ge p(1-t) + (4-t)R_a + t.EE_a(G)$$

Then solving for $EE_a(G)$,

$$EE_a^{\,2}(G) \ge p(1-t) + (4-t)R_a + t.EE_a(G) + p(p-1)$$

$$EE_a^{\,2}(G) \ge p^2 + 4R_a + t\,[EE_a(G) - R_a - p]$$

For $p \ge 2$, the best lower bound for $EE_a(G)$ is attained when $t = 0$.

$$EE_a^{\,2}(G) \ge p^2 + 4R_a$$

$$EE_a(G) \ge \sqrt{p^2 + 4R_a}.$$

which gives the required lower bound for $EE_a(G)$.

From the definition 4.1.1,

$$EE_a(G) = \sum_{k=0}^{\infty}\frac{V_k}{k!} = p + \sum_{k=1}^{\infty}\frac{V_k}{k!}$$

$$EE_a(G) \le p + \sum_{i=1}^{p}\sum_{k\ge 1}\frac{\left(\mu_i^{(a)}\right)^k}{k!}$$

$$EE_a(G) \le p + \sum_{i=1}^{p}\sum_{k\ge 1}\frac{\left|\mu_i^{(a)}\right|^k}{k!}$$

$$\leq p + \sum_{k\geq 1} \frac{1}{k!} \sum_{i=1}^{p} \left(\left({\mu_i}^{(a)} \right)^2 \right)^{\frac{k}{2}} = p + \sum_{k\geq 1} \frac{1}{k!} (2R_a)^{\frac{k}{2}}$$

$$= p - 1 + \sum_{k\geq 0} \frac{\left(\sqrt{2R_a}\right)^k}{k!} = p - 1 + e^{\sqrt{2R_a}}$$

$$EE_a(G) \leq p - 1 + e^{\sqrt{2R_a}}.$$

which gives required upper bound for $EE_a(G)$.

$$\therefore \sqrt{p^2 + 4\mathrm{R}_a} \leq EE_a(G) \leq p - 1 + e^{\sqrt{2\mathrm{R}_a}}.$$

Hence the theorem.

Theorem 4.2.2:

Let G be a connected graph of order p then for $10 \leq a \leq 14$

$$\frac{\left(\sum_{i=1}^{p} e^{\frac{{\mu_i}^{(a)}}{2}}\right)^2 - p}{p-1} \leq EE_a(G) \leq \left(\sum_{i=1}^{p} e^{\frac{{\mu_i}^{(a)}}{2}}\right)^2 - p(p-1).$$

Proof:

We know that, if x_i, $1 \leq i \leq p$ be any real numbers, then

$$p\left[\frac{1}{p}\sum_{i=1}^{p} x_i - \left(\prod_{i=1}^{p} x_i\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} x_i - \left(\sum_{i=1}^{p}\sqrt{x_i}\right)^2 \leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} x_i - \left(\prod_{i=1}^{p} x_i\right)^{\frac{1}{p}}\right]$$

By setting $x_i = e^{{\mu_i}^{(a)}}$ for $i = 1,2,\dots,p$, we have

$$p\left[\frac{1}{p}\sum_{i=1}^{p} e^{{\mu_i}^{(a)}} - \left(\prod_{i=1}^{p} e^{{\mu_i}^{(a)}}\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} e^{{\mu_i}^{(a)}} - \left(\sum_{i=1}^{p}\sqrt{e^{{\mu_i}^{(a)}}}\right)^2 \leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} e^{{\mu_i}^{(a)}} - \left(\prod_{i=1}^{p} e^{{\mu_i}^{(a)}}\right)^{\frac{1}{p}}\right]$$

Consider,

$$p\left[\frac{1}{p}\sum_{i=1}^{p} e^{{\mu_i}^{(a)}} - \left(\prod_{i=1}^{p} e^{{\mu_i}^{(a)}}\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} e^{{\mu_i}^{(a)}} - \left(\sum_{i=1}^{p}\sqrt{e^{{\mu_i}^{(a)}}}\right)^2$$

$$\sum_{i=1}^{p} e^{{\mu_i}^{(a)}} - p\left(e^{\sum_{i=1}^{p}{\mu_i}^{(a)}}\right)^{\frac{1}{p}} \leq p\sum_{i=1}^{p} e^{{\mu_i}^{(a)}} - \left(\sum_{i=1}^{p}\sqrt{e^{{\mu_i}^{(a)}}}\right)^2$$

$$\Rightarrow \left(\sum_{i=1}^{p} e^{\frac{{\mu_i}^{(a)}}{2}}\right)^2 - p \leq (p-1)\sum_{i=1}^{p} e^{{\mu_i}^{(a)}}$$

$$\sum_{i=1}^{p} e^{\mu_i^{(a)}} \geq \frac{\left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p}{p-1}$$

$$EE_a(G) \geq \frac{\left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p}{p-1}.$$

which gives the lower bound for $EE_a(G)$.

Consider,

$$p\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\sum_{i=1}^{p} \sqrt{e^{\mu_i^{(a)}}}\right)^2 \leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\prod_{i=1}^{p} e^{\mu_i^{(a)}}\right)^{\frac{1}{p}}\right]$$

$$p\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\sum_{i=1}^{p} \sqrt{e^{\mu_i^{(a)}}}\right)^2 \leq (p-1)\sum_{i=1}^{p} e^{\mu_i^{(a)}} - p(p-1)\left(e^{\sum_{i=1}^{p}\mu_i^{(a)}}\right)^{\frac{1}{p}}$$

$$\Rightarrow EE_a(G) \leq \left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p(p-1).$$

which gives an upper bound for $EE_a(G)$.

$$\frac{\left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p}{p-1} \leq EE_a(G) \leq \left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p(p-1).$$

Hence the theorem.

4.3. AN UPPER BOUND FOR THE CLOSED NEIGHBOURHOOD DEGREE BASED REACHABILITY ESTRADA INDEX IN TERMS OF THEIR CORRESPONDING ENERGY

In this subsection, we find upper bounds for some closed neighbourhood degree-based reachability Estrada index based on the corresponding closed neighbourhood degree-based reachability Energy.

Theorem 4.3.1:

Let G be a connected graph of diameter not greater than 2 then for $10 \leq a \leq 14$,

$$EE_a(G) - E_a(G) \leq p - 1 - \sqrt{2R_a} + e^{\sqrt{2R_a}}$$

and

$$EE_a(G) \leq p - 1 + e^{E_a(G)}.$$

Proof:

From definition 4.1.1, we have,

$$EE_a(G) = p + \sum_{i=1}^{p}\sum_{k\geq 1}\frac{\left(\mu_i^{(a)}\right)^k}{k!}$$

$$\leq p + \sum_{i=1}^{p}\sum_{k\geq 1}\frac{\left|\mu_i^{(a)}\right|^k}{k!}$$

$$EE_a(G) \leq p + E_a(G) + \sum_{i=1}^{p}\sum_{k\geq 2}\frac{\left|\mu_i^{(a)}\right|^k}{k!}$$

$$EE_a(G) - E_a(G) \leq p - 1 - \sqrt{2R_a} + e^{\sqrt{2R_a}}$$

Another approximation to connect $EE_a(G)$ and $E_a(G)$can be seen as follows:

$$EE_a(G) \leq p + \sum_{i=1}^{p}\sum_{k\geq 1}\frac{\left|\mu_i^{(a)}\right|^k}{k!}$$

$$\leq p + \sum_{k\geq 1}\frac{1}{k!}\left(\sum_{i=1}^{p}\left|\mu_i^{(a)}\right|^k\right)$$

$$\leq p - 1 + \sum_{k\geq 0}\frac{\left(E_a(G)\right)^k}{k!}$$

$$EE_a(G) \leq p - 1 + e^{E_a(G)}.$$

Hence the result.

CONCLUSION:

In this paper, we have introduced some closed neighbourhood degree-based reachability matrices of a graph and obtained their corresponding Energy and Estrada index. We further computed some closed neighbourhood degree-based reachability energy for regular graphs. In addition, we have established bounds for both the Energy and the Estrada index for these matrices.

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