



Compromise solution approach for solving tri-level linear fractional programming problems

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Abstract: In this paper we consider the solution of a tri-level linear fractional programming problems (TLL FPP) by compromise solution approach based on interactive fuzzy goal programming. In this approach optimal solution of the lower-level problem is dependent on the upper-level problem and considers the solution of each DM by randomly pairing up the decision maker (their solutions). Each pair of DMs (solution) give birth to new feasible trial solutions whose features are a random mixture of the features of the solutions of each decision makers. This is in accordance with a hierarchical system where the upper level DM is the main decision maker. The method has no special requirement for the characters of the function and overcome the difficulty discussing the conditions and the algorithms of the optimal solution with the definition of the differentiability of the function. The numerical result shows the proposed algorithm is feasible and efficient, can find global optimal solutions with less computational burden.

Keywords: Tri-level programming problem, optimal solution, compromise solution, satisfactory solution, interactive fuzzy programming, goal Programming.

1. Introduction

A tri-level programming problem (TLPP) consists of three levels, namely, the first level the second level and the third level. The first level decision maker (DM) is called the center. The second level and third level DM are called followers, executes its policies after the decision of higher level DM called leader (center) and then the leader optimizes its objective independently but may be affected by the reaction of the follower i.e. TLPP is a sequence of three optimization problems in which the constraints region of one is determined by the solution of other two. Multi-level programming structure (bi-level or tri-level etc.) [4,6] is used for central economic planning at the regional or national level to create

model problems concerning organizational design, facility location, signal optimization, traffic assignment etc. In a decentralized firm, top management or an executive of headquarters makes a decision such as budget of the firm, and then each division determines production plan in the full knowledge of the budget.

Three-level programming problem (TLPP) can be thought as a static version of the Stackelberg strategy, which is used leader-follower game in which a Stackelberg strategy is used by the leader. Tri-level programming is a class of multi-level programming in which there are three independent decision-makers (DMs). Each DM attempts to optimize its objective function and is affected by the actions of the others DMs. The aim of this paper is to present an interactive fuzzy goal programming approach to determine the preferred compromise solution to a tri-level linear and non-linear programming problems considering the imprecise nature of decision maker's judgments for the objectives. This paper studies a three-level large scale linear fractional programming problem, in which the objective functions at every level are to be maximized. The main advantage of the approach presented here is that the computational load with re-evaluation of the problem again and again by re-defining the elicited membership values of the DMs for searching higher degree of satisfaction does not arise in the solution search process.

A tri-level (or multi-level) organization has following common features[1,2,3,4,5,6,7,8,9,10,11]:

**Interactive decision making units within a predominantly hierarchical structure.*

**Execution of decision is sequential, from upper level to lower level.*

**Each unit independently maximizes or minimizes its own benefits, but is affected by the action of other units through externalities.*

Multi-level programming is a powerful technique for solving hierarchical decision-making problems. Multi-level optimization plays an important role in engineering design, management, and decision making in general. Ultimately, a designer or decision maker needs to make trade offs between disparate and conflicting design objectives. The field of multi-level optimization defines the art and science of making such decisions. The prevailing approach for address this decision-making task is to solve an optimization problem, which yields a candidate solution. A tri-level programming problem (TLPP) is a special case of multi-level programming problem (MLPP). Multi-level programming problem can be defined as a p -person, non-zero sum game with perfect information in which each player moves sequentially from top to bottom. This problem is a nested hierarchical structure. When $p = 3$ we call the system a tri-level programming problem.

Hierarchical optimization or multi-level programming techniques[1,2,3,4,5,6,7,8,9,10,11] are extension of Stackelberg games for solving decentralized planning problem with multiple DMs in a hierarchical organization. The Stackelberg solution has been employed as a solution concept to bi-level programming problems (BLPPs)[1,2,3,7,9] and a considerable number of algorithms for obtaining the solution have been employed. Three-level programming problem (TLPP) can be thought as a static version of the Stackelberg strategy, which is used leader-follower game in which a Stackelberg strategy is used by the leader. When the Stackelberg solution is employed, it is assumed that there is no communication among the DMs, or they do not make any binding agreement even if there exists such communication. However, the above assumption is not always reasonable when we model decision-making problems

in a decentralized firm as a BLPP or TLPP because it is supposed that there exists a cooperative relationship between them.

Stackelberg solution is non-convex programming problem with special structure. From such difficulties a new solution concept, which is easy to compute and reflects the structure of multi-level (or bi-level, tri-level etc.[1,2,3,4,5,6,7,8,9,10,11], programming problem is expected. However a few algorithms have been proposed to solve TLPP, several algorithms have been proposed to BLPP. Recently, as an extension of BLPP the tri-level programming problem has received increasing attention. algorithm are divided into the following classes: global techniques, enumeration methods, transformation methods, meta- heuristic approaches, fuzzy methods, primal-dual interior methods. Also, It has been proven that the MLPPs, especially TLPPs, are NP-Hard problems. The literature shows a few attempts for solving using TLPP. Compromise solution to linear TLPP considering the imprecise nature of the decision makers' judgements for the objectives. This paper studies a three-level large scale linear programming problem, in which the objective functions at every level are to be maximized. We deal with the tri-level programming problem (TLPP) with the essentially co-operative decision makers (DMs) and propose an approach to solve TLPP by weighting method.

In this paper we deal with the TLLFPP with the essentially cooperative DMs and propose an approach to solve TLLFPP. It is a simple method to apply to the tri-level systems compared to the other transformation method. The proposed approach really depends on the configuration of the system, it's over all management and the relative importance of a DM with respect to other DMs in the system. Finally, a numerical example is given to clarify the main results developed in this paper. It was realized that almost every real-life problem involves more than one objective. For such problems, the decision makers have to deal with several objectives conflicting with one another, which are to be optimized simultaneously. For example, in transportation problem, one might like to minimize the operating cost, minimize the average shipping time, minimize the production cost and maximize its capacity. Similarly, in production planning, the plant manager might be interested in obtaining a production programme which would simultaneously maximize profit, minimize the inventory of the finished goods, minimize the overtime and minimize the back orders. Several other problems in modern management can also be identified as having multiple conflicting objectives at different level i.e, multi-level programming problems (MLPP). There is pressing need to develop approaches to solve such type of multi-level linear or non-linear fractional programming problems Decision-making is the process of selecting a possible course of action from all the available alternatives. Many physical problem can be formulated as optimization problem subject to some constraints. Hierarchical systems can be categorized as a multi-level system. It is difficult to define solid optimality for multi-person, decision-making problems. Compromise or co-ordination is usually needed in order to reach a solution, even in a non-cooperative environment. Philosophically, it is also natural to use multiple objective decision making (MODM) methods to model multi person (or two person) decision-making problem if their feasible domain is mutually independent and separable. Most real-world decision problems involve multiple criteria that are

often conflict in general and it is sometimes necessary to conduct trade-off analysis in multiple criteria decision analysis (MCDA).

2. FORMULATION OF TRI-LEVEL PROGRAMMING PROBLEM

Consider a programming problem in which the government is at first level. During the planning period, the government proposes certain goals. In order to optimize the achievement of such goals, it formulates certain policy measures such as taxes and subsidies. The industries at the second level design their course of action keeping such policy measures in mind so that their objectives are fulfilled. The industries supply their products to the consumers in a certain area. The customers at the third level are at liberty to make their purchases from any industries. In doing so, the customers will consider economic criteria such as cost optimization. This is a three level programming problem in which the government's objectives are at least in partial conflict with the two sectors industry and consumers, the policy makers face an optimization problem subject to the optimization problems for industries as well as for the consumers.

Consider a Tri-Level Linear Fractional Programming Problem (TLLFPP) of maximization-type objectives at each level. Mathematically, it can be formulated as follows:

$$\max_{x_1} z_1(\bar{x}) \quad (2.1)$$

$$\max_{x_2} z_2(\bar{x}) \quad (2.2)$$

$$\max_{x_3} z_3(\bar{x}) \quad (2.3)$$

subject to

$$x \in S = \{A_1 x_1 + A_2 x_2 + A_3 x_3 (\geq, =, \leq) B, x \geq 0\}$$

$$\bar{x}_1 \geq 0, \bar{x}_2 \geq 0, x_3(\bar{x}) \geq 0. \quad (2.4)$$

$z_1(\bar{x}) = z_1(x_1, x_2, x_3)$, $z_2(\bar{x}) = z_2(x_1, x_2, x_3)$ and $z_3(\bar{x}) = z_3(x_1, x_2, x_3)$ respectively represent linear fractional objective functions of DM1, DM2, DM3 and $\bar{x}_1 \geq 0, \bar{x}_2 \geq 0, x_3(\bar{x}) \geq 0$ are decision vectors under the control of DM1, DM2 and DM3 respectively.

$A_i x_i (\geq, =, \leq) B$, are linear or non linear constraints.

Let $X =$ set of feasible solutions $= \{ \bar{x} : \bar{x} \in R^n, A_i x_i (\geq, =, \leq) B \}$,

$\bar{x}$ = decision vector in n-dimensional Euclidean space $= x = x_1 \cup x_2 \cup x_3$.

Also let DM1 denote the decision maker at the first level, DM2 denote the decision maker at the second level and DM3 denote the decision maker at the third level.

3. Formulation and solution of tri-level linear fractional programming problem by compromise solution approach

Let a TLLFPP be represented as:

$$\begin{aligned} \max_{x_1} \quad & z_1(\bar{x}) \\ \max_{x_2} \quad & z_2(\bar{x}) \\ \max_{x_3} \quad & z_3(\bar{x}) \end{aligned} \quad (3.1)$$

subject to

$$x \in S = \{A_1 x_1 + A_2 x_2 + A_3 x_3 (\geq, =, \leq) B, x \geq 0\}, \quad \bar{x}_1 \geq 0, \bar{x}_2 \geq 0, \bar{x}_3 \geq 0.$$

Where, $z_1(\bar{x})$, $z_2(\bar{x})$, $z_3(\bar{x})$ and $A_i x_i (\geq, =, \leq) B$, are linear fractional objective functions and linear or non linear constraints respectively. $\bar{x}_1 \geq 0, \bar{x}_2 \geq 0, \bar{x}_3 \geq 0$ are decision vectors under the control of the first level, second level and third level decision maker (DM) respectively.

Let X = set of feasible solutions = $\{\bar{x} : \bar{x} \in R^n, A_i x_i (\geq, =, \leq) B\}$,

$\bar{x}$ = decision vector in n-dimensional Euclidean space = $x = x_1 \cup x_2 \cup x_3$.

In order to consider the imprecise nature of the DMs' judgments for the objectives, assume that the DMs have fuzzy goals for each of the objective functions in the Tri-level linear programming problem, such as : $z_i(\bar{x})$, $i=1,2,3$ should be substantially greater than or equal to some specific value", thus, the objective functions are to be characterized by the associated membership functions. Here, the top level DMs specify fuzzy goals and an aspiration levels to each of them. Fuzzy sets theory has been implemented in mathematical programming since 1970. The basic concepts of fuzzy goals G , fuzzy constraints C , and fuzzy decisions D . Based on these concepts, the fuzzy decision is defined as $D = G \cap C$ which is characterized by the following membership function : $\mu_D(\bar{x}) = \min(\mu_G(x), \mu_C(x))$.

Let us describe the fuzzy goals of TLP problem, assuming that DM_i , $i=1,2,3$ selects the following linear membership function $\mu_i(Z_i(X))$, which is a strictly monotonic increasing function:

$$\mu_i(Z_i(X)) = \begin{cases} 1, & Z_i(X) \geq Z_i^{\max} \\ \frac{Z_i - Z_i^{\min}(X)}{Z_i^{\max} - Z_i^{\min}}, & Z_i^{\min} < Z_i(X) < Z_i^{\max} \\ 0, & Z_i(X) \leq Z_i^{\min} \end{cases} \quad (3.2)$$

Where $Z_i^{\max}(X)$ is the best upper bound and $Z_i^{\min}(X)$ is the worst lower bound of the objective function i , respectively. They are calculated as follows:

$$Z_i^{\max} = \max_x Z_i(X), \text{ s.t. } X \in S \text{ and } Z_i^{\min} = \min_x Z_i(X), \text{ s.t. } X \in S.$$

It is assumed that the first level DM and the second level DM determine the aspiration levels $\hat{F}_l$, $l=1,2$. By using the given linear membership functions and following the fuzzy decision of Bellman and Zadeh[44]. Then, the TLP problem (3.1) can be represented as follows :

$$\max \left\{ \min \mu_i(Z_i(X)) \right\}, \quad i=1,2,3$$

$$X \in S \quad (3.3)$$

By introducing an auxiliary variable λ , problem (3.3) can be transformed into the following linear programming model:

$$\max \lambda \quad (3.4)$$

Subject to, $\mu_1(Z_1(X)) \geq \lambda$, $\mu_2(Z_2(X)) \geq \lambda$, $\mu_3(Z_3(X)) \geq \lambda$, $X \in S$.

The interactive process terminates if the following two conditions are satisfied:

$$1. \mu_1(Z_1(X)) \geq \hat{Z}_1 \text{ and } \mu_2(Z_2(X)) \geq \hat{Z}_2.$$

$$2. \text{ The ratios } \Delta_1 = \frac{\mu_2(Z_2(X))}{\mu_1(Z_1(X))} \text{ and } \Delta_2 = \frac{\mu_3(Z_3(X))}{\mu_2(Z_2(X))} \text{ of satisfactory degrees in the closed interval between its lower and}$$

its upper bounds specified by the first level DM and the second level DM respectively.

Otherwise, for the dissatisfying upper levels DM, the problem (3.4) is re-formulated as an interactive fuzzy goal programming model, let us introduce the following positive and negative deviational variables:

$$Z_l(X) - d_l^+ + d_l^- = F_l,$$

$$d_l^+, d_l^- \geq 0, l=1,2. \quad (3.5)$$

considering over deviation from any fuzzy goal implies the full achievement of the desired values, so the proposed approach assigns only negative deviational variables to the achievement function and minimize negative deviational variables to get a compromise optimal solution. Then equation (3.5) can be written as follows:

$$Z_l(X) + d_l^- \geq F_l, \quad l=1,2. \quad (3.6)$$

Then the membership functions are improved by means of changing the tolerances of the objectives. Such alternative membership function during a solution process reflect the progressive preference. With the improved membership function $\mu_l'(Z_l(X))$ and the constraints described in (3.6), the following problem will be formulated:

$$\begin{aligned} &\max \quad \lambda \\ &\text{Subject to,} \\ &(Z_l(X) + d_l^- \geq F_l), \\ &\mu_l'(Z_l(X)) \geq \lambda, \quad l=1,2. \\ &\mu_3(Z_3(X)) \geq \lambda, \\ &X \in S, \quad d_l^- \geq 0, d_l^- \leq \lambda \end{aligned} \quad (3.7)$$

If an optimal solution to problem (3.7) exists, it follows that the first and the second DMs obtain a satisfactory solution. Then solution procedure of the TLLFPP problem (3.1) can be summarized in the following steps:

4. THE SOLUTION PROCEDURE

Step 1: Develop the TLLFPP as described in problem (3.1).

Step 2: Calculate the individual minimum and maximum of each objective function in the three levels under the given constraints.

Step 3: Ask each DM to determine the best lower bound and the worst upper bound.

Step 4: Define the membership function of each objective function, the initial aspiration levels $1 Z^{\wedge}$, and also the closed intervals for $\Delta_l, l=1,2$.

Step 5: Set $k=1$. Solve the maximum problem (3.4) using MATLAB program for obtaining an optimal solution which maximizes the smaller degree of satisfaction between those of the three DMs.

Step 6: Calculate $\mu_i(Z_i(X^k))$ and $\Delta_i^k = \frac{\mu_{i+1}(Z_{i+1}(X^k))}{\mu_i(Z_i(X^k))}$, $i=1,2,3$

Step 7: The interactive process terminates if $\mu_l(Z_l(X)) \geq \hat{F}_l$, $\Delta_l \in [\Delta_{l1}, \Delta_{l2}]$, $l=1,2$.

Then the upper DMs are satisfied with the optimal solution to problem (3.4), the optimal solution becomes a satisfactory solution. Otherwise, go to step 8.

Step 8: ask the dissatisfying DMs to determine a new aspiration levels.

Step 9: Construct an improved membership functions $\mu_l'(Z_l(X))$ with these new tolerances: $Z_1^{best}(X) \geq \hat{F}_1$ and $Z_1^{worst}(X) \geq F_1^k$

Step 10: Set $k=k+1$ with $\mu_l'(Z_l(X))$ and constraints (3.6), solve problem (3.7) using MATLAB code to get the preferred compromise solution to the TLLFPP. If the current solution (X^k) satisfies the termination conditions and the upper DMs accept it, then the approach stops and the current solution becomes a satisfactory solution. Otherwise, go to step8.

5. NUMERICAL RESULT

In this section we present numerical example to demonstrate the solution procedures by proposed method to solve TLLFPP. The following example considered used to demonstrate the solution procedures and clarify the effectiveness of the proposed approach:

Consider the following TLLFPP

$$\max_{x_1} z_1 = \frac{2x_1 + x_2 + 3x_3 + 3}{x_1 + x_2 + x_3} \quad (\text{First upper level})$$

$$\max_{x_2} z_2 = \frac{x_1 + 4x_2 - 2x_3 + 1}{2x_1 + 2x_2 + x_3 + 3} \quad (\text{Second upper level})$$

$$\max_{x_3} z_2 = \frac{3x_1 + x_2 - x_3 + 2}{4x_1 + 3x_2 - x_3} \quad (\text{Lower Level})$$

subject to

$$2x_1 + x_2 + x_3 \leq 7; \quad x_1 - 2x_2 + 3x_3 \leq 4;$$

$$-x_1 + 2x_2 + 2x_3 \geq 1; \quad x_1 + 2x_2 \geq 3;$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Solution: The individual best (maximal) solution of the objective function $Z_i, i=1,2,3$ subject to the constraints is given in the Table 1.

Table 1: The individual best solutions of the objective functions:

Z_1	Z_2	Z_3
3 at (1.106, 0.947, 1.265)	1.706 at (0, 7, 0)	0.88 at (2, 0.5, 1)

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Z_1	Z_2	Z_3
3 at (1.106, 0.947, 1.265)	1.706 at (0, 7, 0)	0.88 at (2, 0.5, 1)

Suppose that the first and second DM specify the lower and upper bounds of Δ_1 as (0.6, 1.0) and Δ_2 as (0.7, 1) respectively.

Let the lower and upper and upper bounds of Δ_1 as (0.6, 1) and Δ_2 as (0.7, 1)

$$\max \lambda$$

$$\frac{Z_1 - 1}{3 - 1} \geq \lambda$$

$$\frac{Z_2 - 0}{1.7 - 0} \geq \lambda$$

$$\frac{Z_3 - 0}{0.88 - 0} \geq \lambda$$

$$\frac{2x_1 + x_2 + 3x_3 + 3}{x_1 + x_2 + x_3} + d \geq 3$$

$$\frac{x_1 + 4x_2 - 2x_3 + 1}{2x_1 + 2x_2 + x_3 + 3} + e \geq 1.7$$

$$\frac{3x_1 + x_2 - x_3 + 2}{4x_1 + 3x_2 - x_3} + f \geq 0.88$$

$$2x_1 + x_2 + x_3 \leq 7;$$

$$x_1 - 2x_2 + 3x_3 \leq 4;$$

$$-x_1 + 2x_2 + 2x_3 \geq 1;$$

$$x_1 + 2x_2 \geq 3;$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

$$\min = d + e + f$$

$$\frac{Z_1 - 1}{3 - 1} \geq \lambda \quad \frac{Z_2 - 0}{1.7 - 0} \geq \lambda$$

$$\frac{Z_3 - 0}{0.88 - 0} \geq \lambda$$

$$2x_1 + x_2 + x_3 \leq 7;$$

$$x_1 - 2x_2 + 3x_3 \leq 4;$$

$$-x_1 + 2x_2 + 2x_3 \geq 1;$$

$$x_1 + 2x_2 \geq 3;$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \lambda \geq 0.4, \quad .$$



The proposed approach gives the solution as:

x_1	x_2	x_3	z_1	z_2	z_3
2.60	1.23	0.57	2.53	0.78	0.77

6. Conclusion

In this paper we consider the solution of a TLLFPP. Perhaps the most creative task in making a decision is to choose the factors that are important for that decision. Although very few optimization tools are available for TLLFPP, the efficiency of these techniques depends to a great extent on the nature of the mathematical formulation of the problem. Some of these traditional techniques, which give accurate results are computationally expansive and become

inefficient for a large domain. This paper demonstrates the merit of interactive fuzzy goal programming technique in deciding optimal solution of tri-level linear fractional decision-making problem taking into consideration the various constraints and complexities representing the real situation. The main advantage of the proposed method is that the possibility of rejecting the solution again and again by the DM and reevaluation of the problem repeatedly, by redefining the elicited membership functions, needed to reach the satisfactory decision does not arise. An extension of the approach can be extended to solve different MLPPs.

7. Reference

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