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Fractional Order Thermoelastic Analysis of a Penny-Shaped Crack in an Infinite Solid under Transient Thermal Mechanical Loading

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Abstract

We present an analytical fractional-order thermoelastic model for an infinite isotropic solid containing a penny-shaped crack subjected to a combined transient temperature step and axisym- metric mechanical traction. The classical Fourier heat equation is generalized by replacing the first-order time derivative with a Caputo derivative of order $\alpha \in (0, 1]$, enabling the description of thermal-memory and nonlocal diffusion effects relevant to ultra-high-temperature and micro/nano- scale processes.

Keywords: fractional thermoelasticity; Hankel Laplace transform.

1. Introduction

Fourier heat conduction predicts instantaneous propagation and may overestimate peak stresses under ultra-rapid heating. Fractional-order heat conduction, wherein time derivatives are replaced by nonlocal- in-time operators, offers a compact phenomenological route to capture memory effects [7]. Recent works apply fractional models to thermal waves, laser heating, and graded media [3, 9]. In crack mechanics, such nonlocal transport alters the near-tip gradients which driv... Motivated by the author's prior integral-transform analyses of penny-shaped cracks [4] and classical thermoelastic foundations [6, 1, 5], we construct a symbolic solution for a transiently heated circular crack including an optional concurrent mechanical traction on the faces. We derive the Laplace-domain stress intensity factor (SIF) and discuss the influence of the fractional order α on its temporal evolution.

2. Governing equations with fractional order

Let (r, θ, z) be cylindrical coordinates with the crack in the plane $z = 0$ and occupying $r < a$. The fractional heat equation with Caputo derivative ${}^{C}D_t^{\alpha}$ reads

$$ {}^{C}D_t^{\alpha} T = \kappa \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right) - \frac{T_0 \alpha_t}{\rho c} \frac{\partial}{\partial t} (\nabla \cdot \boldsymbol{u}), \tag{1} $$

where $K = k/(\rho c)$ is the thermal diffusivity and α_t the thermal expansion coefficient (to distinguish from the fractional order α). The Caputo derivative of order $\alpha \in (0, 1]$ is defined by [7]

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{f}(\tau)}{(t-\tau)^{\alpha}} \, d\tau. \tag{2} $$

The constitutive and equilibrium relations follow classical linear thermoelasticity:

$$ \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - 3k\alpha_t T\delta_{ij}, \tag{3} $$

$$\sigma_{ij,j} = 0, \qquad \varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \tag{4}$$

3. Loading and boundary conditions

Initial state is stress-free and isothermal: $T(\cdot, 0) = 0, u(\cdot, 0) = \mathbf{0}$. A thermal step $\Delta T H(t)$ is applied on the crack faces ($r < a, z = 0$), possibly accompanied by an axisymmetric traction $p_0 H(t)$:

$$T(r, 0, t) = \Delta T\, H(t), \qquad 0 < r < a, \tag{5}$$

$$\sigma_{zz}(r, 0, t) = -p_0 H(t), \quad \sigma_{rz}(r, 0, t) = 0, \qquad 0 < r < a. \tag{6}$$

For $r > a$, the intact plane enforces continuity of temperature and heat flux; all fields decay as $r, |z| \to \infty$.

4. Transform solution: Hankel in r and Laplace in t

Define the zeroth-order Hankel and Laplace transforms

$$\widehat{f}(s,\xi,z) = \int_0^\infty r J_0(\xi r)\left[\int_0^\infty e^{-st} f(r,z,t)\, dt\right] dr, \tag{7}$$

$$f(r,z,t) = \int_0^\infty \xi J_0(\xi r)\left[\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\widehat{f}(s,\xi,z)\, ds\right] d\xi. \tag{8}$$

Using $L\{{}^C D_t^\alpha f\} = s^\alpha \bar{f} - s^{\alpha-1} f|_{t=0}$ and the zero initial condition, the transformed heat equation becomes

$$s^\alpha \widehat{T} = \kappa\left(\frac{\partial^2 \widehat{T}}{\partial z^2} - \xi^2 \widehat{T}\right) - \frac{T_0 \alpha_t s}{\rho c}\frac{\partial \widehat{u}_z}{\partial z}. \tag{9}$$

The decaying solution in $|z|$ is

$$\bar{T}(s,\xi,z) = \widehat{T}_0(s,\xi) e^{-\beta|z|}, \qquad \beta = \sqrt{\xi^2 + \frac{s^\alpha}{\kappa}}. \tag{10}$$

The crack-plane temperature amplitude $\bar{T}_0$ follows from the imposed surface temperature.

4.1 Thermoelastic potentials and stresses

Introduce a displacement potential $\bar{\varphi}$ so that $\bar{U}_r = \partial_r \bar{\varphi}$, $\bar{U}_z = \partial_z \bar{\varphi}$. Equilibrium yields

$$\frac{d^2\widehat{\phi}}{dz^2} - \xi^2 \widehat{\phi} = \frac{3K\alpha_t(1-2\nu)}{E}\widehat{T}(s,\xi,z). \tag{11}$$

Solving with decay in $|z|$ gives

$$\widehat{\phi}(s,\xi,z) = C(s,\xi)e^{-\xi|z|} + \frac{3K\alpha_t(1-2\nu)}{E(\xi^2 - \beta^2)}\widehat{T}_0(s,\xi)e^{-\beta|z|}. \tag{12}$$

Hence the normal traction on the plane $z = 0$ in the Laplace–Hankel domain can be written as

$$\widehat{\sigma}_{zz}(s, \xi, 0) = A(s, \xi; \alpha)\widehat{T}_0(s, \xi) + P(s, \xi)\widehat{p}_0(s), \tag{13}$$

where $p_0(s) = p_0/s$ and the kernels A, P are explicit functions of (E, ν, α_t, k) and $\beta(s, \xi)$ from (10).

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5. Dual-integral system and SIF

Imposing traction-free conditions on the crack faces (after subtracting the applied p_0) and continuity for $r > a$, we obtain the classical dual-integral system [8, 2]. Its solution yields the Laplace-domain mode-I SIF

$$\widehat{K}_I(s;\alpha) = \int_0^\infty \Phi(s,\xi;a,\alpha)\,\widehat{T}_0(s,\xi)\,d\xi + \int_0^\infty \Psi(s,\xi;a)\,\widehat{p}_0(s)\,d\xi, \qquad (14)$$

where Φ and Ψ are edge kernels determined by the crack radius a and the operator factors in (13). The time-domain SIF follows via inverse Laplace transform: $K_I(t;\alpha) = \mathcal{L}^{-1}\{\widehat{K}_I(s;\alpha)\}(t)$.

6. Asymptotics and limiting cases

Short-time behavior ($t \to 0^+$) is governed by large-s asymptotics; from $\beta \sim (s_\alpha/k)^{1/2}$ one infers a weakened singularity compared to the classical $\alpha = 1$ case. Long times ($t \to \infty$) correspond to small s and the SIF decays as the thermal field relaxes. Limiting models:

- $\alpha \to 1$: recovers Fourier thermoelasticity [6,5].
- $\alpha \to 0^+$: strong-memory limit with highly damped thermal front [7].

7. Non-dimensionalization

Let a be the length scale and $\tau = a^2/k$ the diffusive time. Normalize

$$\tilde{r} = \frac{r}{a}, \quad \tilde{t} = \frac{t}{\tau}, \quad \tilde{\xi} = \xi a, \quad \tilde{s} = s\tau, \quad \tilde{\beta} = \beta a = \sqrt{\tilde{\xi}^2 + \tilde{s}^\alpha}. \qquad (15)$$

Scale stresses by $\sigma_0 = E\alpha_t \Delta T/(1-\nu^2)$ and define $\widetilde{K}_I = K_I/(\sigma_0\sqrt{\pi a})$. Then (14) reads symbolically

$$\widetilde{K}_I(\tilde{t};\alpha) = \mathcal{L}^{-1}\left\{\int_0^\infty \widetilde{\Phi}(\tilde{s},\tilde{\xi};\alpha)\,d\tilde{\xi} + \int_0^\infty \widetilde{\Psi}(\tilde{s},\tilde{\xi})\,\frac{p_0}{\sigma_0}\,\frac{d\tilde{\xi}}{\tilde{s}}\right\}(\tilde{t}). \qquad (16)$$

8. Physical interpretation

Fractional order α encodes heat-memory: smaller α delays the thermal front, flattening near-tip gradients and reducing the initial SIF peak. The analytical structure shows that the Laplace-domain kernel replaces s with s_α in the thermal operator, which shifts the dominant balance in early-time asymptotics. This aligns with recent experimental and modeling observations for ultrafast heating of advanced materials [3,9].

9. Schematic of geometry

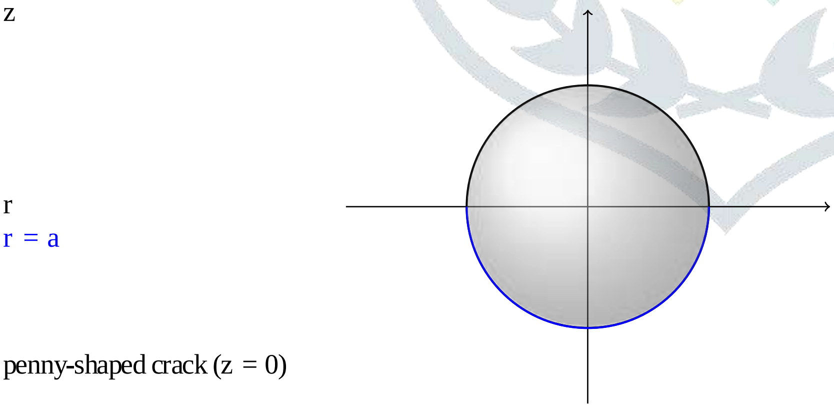

Figure 1: Infinite isotropic solid with a penny-shaped crack of radius a in the plane z = 0.

10. Conclusions

We proposed a fractional-order thermoelastic formulation for a penny-shaped crack subjected to transient thermal–mechanical loading. By incorporating a Caputo derivative of order α, we obtained symbolic Hankel–Laplace solutions and a compact Laplace-domain representation for the SIF. The theory recovers the classical model for $\alpha \to 1$ and suggests reduced early-time stress concentrations for $\alpha < 1$, contributing a rigorous basis for analyzing ultra-high-temperature and ultrafast-heat...

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