A THEOREM ON BILINEAR GENERATING FUNCTIONS

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Abstract

In this paper, we have proved a theorem on bilinear generating functions for a set of polynomials, satisfy certain conditions. The theorem has been used to obtain new bilinear generating functions for some polynomials. It is also note that some of the results given by Srivastava and Singhal ([7], 1972), follow easily as its special cases.

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1 Introduction

Generating functions play a large role in the study of special functions. Various methods have been used by researcher in the derivation of generating functions of special functions. In the present paper property of differential operator has been adopted to obtain some novel results of generating functions involving sequence of functions $f_n(x)$ and $f_n(y)$ defined by

$$f_n(x) = \lambda_1(n)\, G(x) x^{-k_1 n}\, \theta_1^n[g(x)], \qquad \theta_1 = x^{k_1+1} \frac{d}{dx} \qquad \dots(1.1)$$

$$f_n(y) = \lambda_2(n)\, G(y) y^{-k_2 n}\, \theta_2^n[g(y)], \qquad \theta_2 = y^{k_2+1} \frac{d}{dy}, \qquad \dots(1.2)$$

where $g(x),\ g(y)$ and $G(x),\ G(y)$ are independent of n.

The main object of the present paper is to derive some bilinear generating relations by using the property of differential operator.

2 Main Result

Theorem 2.1: *If there exists a bilinear generating relation*

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n f_n(x) f_n(y) t^n, \qquad \qquad \dots(2.1)$$

where $a_n \neq 0$ are arbitrary constant, $f_n(x)$ and $f_n(y)$ are defined in (1.1) & (1.2) then

$$G(x)\, G(y)\, F\left[\frac{x}{(1-x^{k_1}t)^{\frac{1}{k_1}}},\ \frac{y}{(1-y^{k_2}t)^{\frac{1}{k_2}}},\ \frac{x^{k_1}y^{k_2}\,wt}{k(1-x^{k_1}y^{k_2}t)^{\frac{1}{k_2}}}\right] \Big/ G\left[\frac{x}{(1-x^{k_1}t)^{\frac{1}{k_1}}}\right] G\left[\frac{y}{(1-y^{k_2}t)^{\frac{1}{k_2}}}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\sigma_m(w)\, f_m(x)\, f_m(y)}{\lambda_1(m)\, \lambda_2(m)}\ \frac{x^{k_1 m}\, x^{k_2 m}}{m!}\ \left(\frac{t}{k}\right)^m, \qquad \qquad \dots(2.2)$$

where $\qquad \sigma_m(w) = \sum_{n=0}^{m} (-m)_n\, a_n\, \lambda_1(m)\, \lambda_2(m)\, (-w)^n. \qquad \qquad \dots(2.3)$

Proof:

Let us consider the bilinear generating function:

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n f_n(x) f_n(y) t^n.$$

Replacing t by $x^{k_1} y^{k_2} zw$ in above function and using (1.1) and (1.2), we get

$$F(x,y,x^{k_1} y^{k_2} zw) = \sum_{n=0}^{\infty} a_n\, \lambda_1(n)\, \lambda_2(n)\, G(x)\, G(y)\, z^n w^n\, \theta_1^n[g(x)]\, \theta_2^n[g(y)]. \qquad \dots(2.4)$$

Operating $e^{z\,\theta_1\theta_2}$ in both the sides of (2.4) and using the property of the differential operator given by

$$e^{z\,\theta_1\theta_2} f(x,y) = f[x\,(1-x^{k_1}zk)^{-\frac{1}{k_1}},\ y\,(1-y^{k_2}zk)^{-\frac{1}{k_2}}, \qquad \dots(2.5)$$

we get

$$F\left[\frac{x}{(1-x^{k_1}zk)^{\frac{1}{k_1}}},\ \frac{y}{(1-y^{k_2}zk)^{\frac{1}{k_2}}},\ \frac{x^{k_1}y^{k_2}\,zw}{(1-x^{k_1}y^{k_2}\,zk)}\right] = \sum_{n=0}^{\infty} a_n\, \lambda_1(n)\, \lambda_2(n)\, G\left[\frac{x}{(1-x^{k_1}\,zk)^{\frac{1}{k_1}}}\right]$$

$$G\left[\frac{y}{(1-y^{k_2}\,zk)^{\frac{1}{k_2}}}\right] z^n w^n\, e^{z\,\theta_1\theta_2}\, \theta_1^n[g(x)]\, \theta_2^n[g(y)]$$

$$\Rightarrow F\left[\frac{x}{(1-x^{k_1}zk)^{\frac{1}{k_1}}},\ \frac{y}{(1-y^{k_2}zk)^{\frac{1}{k_2}}},\ \frac{x^{k_1}y^{k_2}\,zw}{(1-x^{k_1}y^{k_2}\,zk)}\right] = \sum_{n=0}^{\infty} a_n\, \lambda_1(n)\, \lambda_2(n)\, G\left[\frac{x}{(1-x^{k_1}zk)^{\frac{1}{k_1}}}\right]$$

$$G\left[\frac{y}{(1-y^{k_2}zk)^{\frac{1}{k_2}}}\right] (zw)^n\, \sum_{m=0}^{\infty} \frac{(z\,\theta_1\theta_2)^m}{m!}\, \theta_1^n[g(x)]\, \theta_2^n[g(y)]$$

$$\Rightarrow F\left[\frac{x}{(1-x^{k_1}zk)^{\frac{1}{k_1}}},\ \frac{y}{(1-y^{k_2}zk)^{\frac{1}{k_2}}},\ \frac{x^{k_1}y^{k_2}\,zw}{(1-x^{k_1}y^{k_2}\,zk)}\right] \Big/ G\left[\frac{x}{((1-x^{k_1}zk))^{\frac{1}{k_1}}}\right] G\left[\frac{y}{(1-y^{k_2}zk)^{\frac{1}{k_2}}}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{a_n\, \lambda_1(n)\, \lambda_2(n)\, w^n\, z^{m+n}}{m!}\, \theta_1^{m+n}[g(x)]\, \theta_2^{m+n}[g(y)]$$

$$= \sum_{m=0}^{\infty} \frac{\sigma_m(w) z^m\, \theta_1^m[g(x)]\, \theta_2^m[g(y)]}{m!}. \qquad \qquad \dots(2.6)$$

Putting the values of $\theta_1^m[g(x)]\, \theta_2^m[g(y)]$ from (1.1) and (1.2) in (2.6), we get

$$F\left[\frac{x}{(1-x^{k_1}\,zk)^{\frac{1}{k_1}}},\ \frac{y}{(1-y^{k_2}\,zk)^{\frac{1}{k_2}}},\ \frac{x^{k_1}\,y^{k_2}\,zw}{(1-x^{k_1}y^{k_2}\,zk)}\right] \Big/ G\left[\frac{x}{((1-x^{k_1}\,zk))^{\frac{1}{k_1}}}\right] G\left[\frac{y}{(1-y^{k_2}\,zk)^{\frac{1}{k_2}}}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\sigma_m(w) z^m}{m!}\ \frac{f_m(x)\, f_m(y)\, x^{k_1 m}\, y^{k_2 m}}{\lambda_1(m)\, \lambda_2(m)\, G(x)\, G(y)}.$$

Further simplifying, we get

$$G(x)\, G(y)\, F\left[x\,(1 - x^{k_1}zk)^{-\frac{1}{k_1}},\ y\,(1 - y^{k_2}zk)^{\frac{1}{k_2}},\ x^{k_1}\, y^{k_2}\, zw\,(1 - x^{k_1}y^{k_2}\,zk)^{-1}\right] /$$

$$G\left[\frac{x}{(1 - x^{k_1}\,zk)^{\frac{1}{k_1}}}\right] G\left[\frac{y}{(1 - y^{k_2}\,zk)^{\frac{1}{k_2}}}\right] = \sum_{m=0}^{\infty} \frac{\sigma_m(w)\, f_m(x)\, f_m(y)\, x^{k_1 m}\, y^{k_2 m}}{\lambda_1(m)\, \lambda_2(m)\, m!}\, z^m. \qquad \ldots(2.7)$$

Hence the theorem follows on replacing zk by t.

3 Applications

On replacing $k_2 = 1$ & t by $x^{-k_1}\, t$ in (2.2), the theorem can be written as

$$G(x)\, G(y)\, F\left[\frac{x}{(1-t)^{\frac{1}{k_1}}},\ \frac{y}{1-yt},\ \frac{ywt}{k\,(1-y\,t)}\right] / G\left[\frac{x}{(1-t)^{\frac{1}{k_1}}}\right] G\left[\frac{y}{1-yt}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\sigma_m(w)\, f_m(x)\, f_m(y)}{\lambda_1(m)\, \lambda_2(m)\, m!}\, \left(\frac{yt}{k_1}\right)^m, \qquad \ldots(3.1)$$

where $\sigma_m(w)$ is defined in (2.3).

(i) the first application of this theorem, we recall the following known formula for the Kounhauser biorthogonal polynomials $Y_n^\alpha(x;\ k_1)$ and $Y_n^\beta(y;\ k_2)$ as

$$Y_n^\alpha(x;\ k_1) = \frac{x^{-\alpha - k_1 n - 1}}{k_1^n\, n!}\, e^x\, \theta_1^n(x^{\alpha+1}\, e^{-x}), \qquad \ldots(3.2)$$

$$Y_n^\beta(y;\ k_2) = \frac{y^{-\beta - k_2 n - 1}}{k_2^n\, n!}\, e^y\, \theta_2^n(y^{\beta+1}\, e^{-y}) \qquad \ldots(3.3)$$

where $\alpha > -1$ and k_1 & k_2 are positive integer.

Formulae (3.2) and (3.3) are of type (1.1) and (1.2) with $\lambda_1(n) = \frac{1}{k_1^n\, n!}$, $\lambda_2(n) = \frac{1}{k_2^n\, n!}$,

$G(x) = x^{-\alpha-1}\, e^{-x}$, $G(y) = y^{-\beta-1}\, e^{-y}$ and $g(x) = x^{\alpha+1}\, e^{-x}$, $g(y) = y^{\beta+1}\, e^{-y}$.

Thus the application of our theorem to these polynomials $Y_n^\alpha(x;\ k_1)$ and $Y_n^\beta(y;\ k_2)$ provides us with.

Corollary 3.1:

If $\qquad\qquad F(x,y,t) = \sum_{n=0}^{\infty} a_n\, Y_n^\alpha(x;\ k_1)\, Y_n^\beta(y;\ k_2)\, t^n, \qquad \ldots(3.4)$

then

$$(1-t)^{\frac{(-\alpha-1)}{k_1}}\, (1-t)^{\frac{(-\beta-1)}{k_2}}\ \ \exp\left[x\left\{1 - (1-t)^{\frac{-1}{k_1}}\right\}\right]\ \exp\left[y\left\{1 - (1-t)^{\frac{-1}{k_2}}\right\}\right]$$

$$F\left[\frac{x}{(1-t)^{\frac{1}{k_1}}},\ \frac{y}{1-yt},\ \frac{wyt}{k\,(1-yt)}\right] = \sum_{n=0}^{\infty} \sigma_n(w)\, Y_n^\alpha(x;\ k_1)\, Y_n^\beta(y;\ k_2)\, t^n, \qquad \ldots(3.5)$$

where $\sigma_n(w) = \sum_{m=0}^{n}\, a_m\, \binom{n}{m}\, \left(\frac{yt}{k}\right)^m$.

Since $Y_n^\alpha(x;\ 1) = L_n^\alpha(x)$ and $Y_n^\beta(y;\ 1) = L_n^\beta(y)$, $n = 0, 1, 2, \ldots$, for $k_1 = 1$, $k_2 = 1$, the formula (3.5) leads us to the result obtained by Srivastava and Singhal ([7] (1972), p. 757) which is the corrected version of a result proved by Al- Salam ([1] (1964), p.134).

(ii) The polynomials $G_n^\alpha(x, r, p, k_1)$ and $G_n^\beta(y, s, q, k_2)$ defined by Srivastava and Singhal ([6] (1971), p.75) as follows:

$$G_n^\alpha(x, r, p, k_1) = \frac{x^{-\alpha - k_1 n}}{n!} e^{px^r} \theta_1^n(x^\alpha e^{-px^r}), \qquad \dots(3.6)$$

$$G_n^\beta(y, s, q, k_2) = \frac{y^{-\beta - k_2 n}}{n!} e^{qy^s} \theta_2^n(y^\beta e^{-qy^s}). \qquad \dots(3.7)$$

These are also of type (1.1) and (1.2), with $\lambda_1(n) = \frac{1}{n!} = \lambda_2(n)$, $G(x) = x^{-\alpha} e^{px^r}$,

$G(y) = y^{-\beta} e^{qy^s}$ and $g(x) = x^\alpha e^{-px^r}$, $g(y) = y^\beta e^{-qy^s}$ and therefore we arrive at the following corollary:

Corollary 3.2:

If $\qquad F(x, y, t) = \sum_{n=0}^{\infty} a_n \ G_n^\alpha(x, r, p, k_1) G_n^\beta(y, s, q, k_2) t^n \qquad \dots(3.8)$

then

$$(1-t)^{\frac{-\alpha}{k_1}} (1-t)^{\frac{-\beta}{k_2}} \quad \exp\left[e^{px^r}\left\{1 - (1-t)^{\frac{-r}{k_1}}\right\}\right] \exp\left[e^{qy^s}\left\{1 - (1-t)^{\frac{-s}{k_2}}\right\}\right]$$

$$F\left[\frac{x}{(1-t)^{\frac{1}{k_1}}}, \ \frac{y}{(1-y^{k_2} t)^{\frac{1}{k_2}}}, \ \frac{x^{k_1} y^{k_2} wt}{k(1-y^{k_2} t)}\right] = \sum_{n=0}^{\infty} \sigma_n(w) \ G_n^\alpha(x, r, p, k_1) G_n^\beta(y, s, q, k_2) t^n, \quad \dots(3.9)$$

where $\quad \sigma_n(w) = \sum_{m=0}^{n} a_m \binom{n}{m} \left(\frac{yt}{k}\right)^m,$

which is as parallel to Srivastava and Singhal given in ([6] (1971))

4 Conclusion

In this work, we have successfully proved a general theorem on bilinear generating functions for a class of polynomials satisfying specific conditions. This theorem not only extends the scope of generating function techniques but also provides a unified framework for deriving new bilinear generating functions for various families of polynomials. Furthermore, the theorem encompasses several previously known results, including those by Srivastava and Singhal ([7], 1972) as particular cases, thereby demonstrating its broad applicability and significance.

5 References

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