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On Properties and Applications of Two Dimensional Linear Canonical Transform

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Abstarct:

This paper studies the two-dimensional Linear Canonical Transform (2D-LCT) and establishes its key properties, including linearity, shifting, scaling, modulation, and axis inversion. These results extend the operational framework of the LCT to two dimensions. The applicability of the 2D-LCT is demonstrated by solving the two-dimensional wave equation using the transform. The findings highlight the effectiveness of the 2D-LCT in analyzing two-dimensional signals and solving partial differential equations.

Keywords: Linear canonical transform, wave equation, integral transform.

1. Introduction

The Linear Canonical Transform (LCT)[1,2,3] is a widely studied integral transform that generalizes several classical transforms, including the Fourier, fractional Fourier, and Laplace transforms. Because of its rich parameterization and structural flexibility, the LCT has become an essential analytical tool in optics, signal processing, time–frequency analysis, and the modelling of physical systems [4,5]. While the one-dimensional LCT [6,7,8] has been extensively explored, many real-world problems—such as image processing, beam propagation, and multidimensional wave dynamics—naturally require a two-dimensional formulation. The two-dimensional Linear Canonical Transform (2D-LCT) extends the capabilities of the one-dimensional case by providing a more versatile representation of multidimensional signals [9]. Its parameterized kernel enables various geometric transformations, making it suitable for analysing spatially varying structures and solving multidimensional differential equations. Despite its broad potential, the operational properties of the 2D-LCT have not been as widely documented as those of its one-dimensional counterpart.

In this paper, we investigate fundamental properties of the 2D-LCT, including linearity, shifting, scaling, modulation, and axis inversion. Establishing these properties strengthens the theoretical foundation of the transform and facilitates its use in practical applications. To demonstrate the analytical power of the 2D-LCT, we apply it to the two-dimensional wave equation and obtain its solution in the transform domain. The results presented here contribute to a deeper understanding of the 2D-LCT and highlight its effectiveness in handling multidimensional signal transformations and partial differential equations. This work aims to support further research in areas where multidimensional transforms play a critical role, particularly in optical systems, mathematical physics, and advanced signal processing.

2. *Definition:* 2D-LCT

For any matrix $A_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} \in SL(2,R)$, $b_j \neq 0, j = 1,2$, the two dimensional linear canonical transform of a function $f(x_1,x_2) \in L^1(R^2)$ is defined by

$$L_{A_1,A_2}\{f(x_1,x_2)\}(u_1,u_2) = F_{A_1,A_2}(u_1,u_2) = \int_{R^2} f(x_1,x_2)\, K_{A_1,A_2}(x_1,x_2,u_1,u_2)dx_1dx_2, (2.1)$$

where

$$K_{A_1,A_2}(x_1,x_2,u_1,u_2) = \prod_{j=1}^{2} K_{A_j}(x_j,u_j) = \prod_{j=1}^{2} \sqrt{\frac{1}{2i\pi b_j}}\, e^{\frac{i}{2}\left(\frac{a_j}{b_j}x_j^2 - \frac{2}{b_j}x_ju_j + \frac{d_j}{b_j}u_j^2\right)}$$

Theorem 1: (Linearity) If $f_1(x_1,x_2), f_2(x_1,x_2) \in L^1(R^2)$ and α_1, α_2 are any two complex constants, then

$$L_{A_1,A_2}\left\{\sum_{\mu=1}^{2} \alpha_\mu f_\mu\,(x_1,x_2)\right\}(u_1,u_2) = \sum_{\mu=1}^{2} \alpha_\mu L_{A_1,A_2}\,\{f_\mu(x_1,x_2)\}(u_1,u_2)$$

Proof. The proof follows readily from Definition (2.1) and is left for the reader.

Theorem 2: If $f(x_1,x_2) \in L^1(R^2)$, then the following results hold:

a) (Shifting) $L_{A_1,A_2}\{f(x_1-\beta_1, x_2-\beta_2)\}(u_1,u_2)$

$$= \left(\prod_{j=1}^{2} e^{i\left(c_j\beta_ju_j - \frac{1}{2}c_ja_j\beta_j^2\right)}\right) F_{A_1,A_2}(u_1 - a_1\beta_1, u_2 - a_2\beta_2)$$

b) (Scaling) $L_{A_1,A_2}\{f(\beta_1x_1,\beta_2x_2)\}(u_1,u_2) = \frac{1}{\beta_1\beta_2} F_{A_1^*,A_2^*}(u_1\beta_1, u_2\beta_2)$,

where $A_j^* = \begin{bmatrix} a_j & b_j\beta_j^2 \\ \frac{c_j}{\beta_j^2} & d_j \end{bmatrix}, j = 1,2.$

c) (Modulation)

$L_{A_1,A_2}\{e^{i(\beta_1x_1+\beta_2x_2)}f(x_1,x_2)\}(u_1,u_2)$

$$= e^{id_1\beta_1u_1 - \frac{1}{2}b_1d_1\beta_1^2} e^{id_2\beta_2u_2 - \frac{1}{2}b_2d_2\beta_2^2} F_{A_1,A_2}(u_1 - b_1\beta_1, u_2 - b_2\beta_2)$$

d) Axis Inversion $L_{A_1,A_2}\{f(-x_1,-x_2)\}(u_1,u_2) = F_{A_1,A_2}(-u_1,-u_2)$

Proof. **(a)** By definition (2.1), we have

$$L_{A_1,A_2}\{f(x_1-\beta_1,x_2-\beta_2)\}(u_1,u_2)$$

$$=\int_{R^2} f(x_1-\beta_1,x_2-\beta_2)\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}e^{\frac{i}{2}\left(\frac{a_j}{b_j}x_j^2-\frac{2}{b_j}x_ju_j+\frac{d_j}{b_j}u_j^2\right)}dx_1dx_2$$

Substituting $x_j-\beta_j=y_j, j=1,2$, above gives

$$=\int_{R^2} f(y_1,y_2)\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}e^{\frac{i}{2}\left(\frac{a_j}{b_j}(y_j+\beta_j)^2-\frac{2}{b_j}(y_j+\beta_j)u_j+\frac{d_j}{b_j}u_j^2\right)}dy_1dy_2$$

$$=\prod_{j=1}^{2}e^{\frac{-i}{2}\left(\frac{a_jd_j-1}{b_j}\right)a_j\beta_j^2+i\left(\frac{a_jd_j-1}{b_j}\right)\beta_ju_j}$$

$$\int_{R^2} f(y_1,y_2)\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}e^{\frac{i}{2}\left(\frac{a_j}{b_j}y_j^2-\frac{2}{b_j}y_j(u_j-a_j\beta_j)+\frac{d_j}{b_j}(u_j-a_j\beta_j)^2\right)}dy_1dy_2$$

$$=\left(\prod_{j=1}^{2}e^{i\left(c_j\beta_ju_j-\frac{1}{2}c_ja_j\beta_j^2\right)}\right)F_{A_1,A_2}(u_1-a_1\beta_1,u_2-a_2\beta_2)$$

(b) $L_{A_1,A_2}\{f(\beta_1x_1,\beta_2x_2)\}(u_1,u_2)$

$$=\int_{R^2} f(\beta_1x_1,\beta_2x_2)\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}e^{\frac{i}{2}\left(\frac{a_j}{b_j}x_j^2-\frac{2}{b_j}x_ju_j+\frac{d_j}{b_j}u_j^2\right)}dx_1dx_2$$

On substituting $\beta_jx_j=y_j, j=1,2$

$$=\int_{R^2} f(y_1,y_2)\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}e^{\frac{i}{2}\left(\frac{a_j}{b_j}\left(\frac{y_j}{\beta_j}\right)^2-\frac{2}{b_j}\left(\frac{y_j}{\beta_j}\right)u_j+\frac{d_j}{b_j}u_j^2\right)}\frac{dy_1}{\beta_1}\frac{dy_2}{\beta_2}$$

$$=\frac{1}{\beta_1\beta_2}\int_{R^2} f(y_1,y_2)\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}e^{\frac{i}{2}\left(\frac{a_j}{b_j\beta_j^2}y_j^2-\frac{2}{b_j\beta_j^2}x_j(\beta_ju_j)+\frac{d_j}{b_j\beta_j^2}(\beta_ju_j)^2\right)}dy_1dy_2$$

$$=\frac{1}{\beta_1\beta_2}F_{A_1^*,A_2^*}(u_1\beta_1,u_2\beta_2)$$

where $A_j^*=\begin{bmatrix}a_j & b_j\beta_j^2\\ \frac{c_j}{\beta_j^2} & d_j\end{bmatrix}, j=1,2.$

The proofs of (c)-(d) can be easily obtained similar to one dimensional case.

3. **Definition: Schwartz space $S(R^2)$:** The so-called space of smooth functions of rapid descent $S(R^2)$ is defined by a collection of complex valued functions

$$S(R^2) = \left\{\phi(x_1, x_2) \in C^\infty(R^2): \sup_{(x_1,x_2)\in R^2} \left|\prod_{j=1}^{2} x_j^{\alpha_j} \left(\frac{\partial}{\partial x_j}\right)^{\beta_j} \phi(x_1, x_2)\right| < \infty\right\},$$

$\forall \alpha_j, \beta_j \in N, j = 1,2.$

The generalization of the Schwartz space in the two-dimensional linear canonical transform (2D-LCT) domain is defined as follows.

4. **Definition: Schwartz space $S_{A_1,A_2}(R^2)$:** The general function space $S_{A_1,A_2}(R^2)$ is defined by a collection of a complex valued functions

$$S_{A_1,A_2}(R^2) = \left\{\phi(x_1, x_2) \in C^\infty(R^2): \sup_{(x_1,x_2)\in R^2} \left|\prod_{j=1}^{2} x_j^{\alpha_j} \left(\Lambda_{x_j}\right)^{\beta_j} \phi(x_1, x_2)\right| < \infty\right\},$$

$\forall \alpha_j, \beta_j \in N, j = 1,2$, where $\Lambda_{x_j} = \frac{\partial}{\partial x_j} - i\frac{a_j}{b_j}x_j$.

This generalized space extends the classical Schwartz space, accommodating the broader functional framework required for analysis within the 2D-LCT domain. It inherits properties of rapid decay and smoothness, tailored to the transformed coordinate system imposed by the 2D-LCT.

Proposition 1: Let $\prod_{j=1}^{2} K_{A_j}(x_j, u_j)$ be the 2D-LCT kernel and $\Lambda_{x_j} = \frac{\partial}{\partial x_j} - i\frac{a_j}{b_j}x_j, j = 1,2$, then

(i) $\Lambda_{x_1}^n \left\{\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\right\} = \left(-i\frac{u_1}{b_1}\right)^n \prod_{j=1}^{2} K_{A_j}(x_j, u_j), \forall n \in N$

(ii) $\Lambda_{x_2}^n \left\{\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\right\} = \left(-i\frac{u_2}{b_2}\right)^n \prod_{j=1}^{2} K_{A_j}(x_j, u_j), \forall n \in N$

(iii) $L_{A_1,A_2}\left\{\overline{\Lambda}_{x_1}^n \phi(x_1, x_2)\right\}(u_1, u_2) = \left(-i\frac{u_1}{b_1}\right)^n L_{A_1,A_2}\{\phi(x_1, x_2)\}(u_1, u_2)$, $\forall n \in N$ and $\phi(x_1, x_2) \in S_{A_1,A_2}(R^2)$

(iv) $L_{A_1,A_2}\left\{\overline{\Lambda}_{x_2}^n \phi(x_1, x_2)\right\}(u_1, u_2) = \left(-i\frac{u_2}{b_2}\right)^n L_{A_1,A_2}\{\phi(x_1, x_2)\}(u_1, u_2)$, $\forall n \in N$ and $\phi(x_1, x_2) \in S_{A_1,A_2}(R^2)$,

where $\overline{\Lambda}_{x_j} = -\frac{\partial}{\partial x_j} - i\frac{a_j}{b_j}x_j, j = 1,2.$

Proof.

(i) We have

$$\Lambda_{x_1}\left\{\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\right\} = \left(\frac{\partial}{\partial x_1} - i\frac{a_1}{b_1}x_1\right)\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}\, e^{\frac{i}{2}\left(\frac{a_j}{b_j}x_j^2 - \frac{2}{b_j}x_ju_j + \frac{d_j}{b_j}u_j^2\right)}$$

$$= \left\{i\left(\frac{a_1}{b_1}x_1 - \frac{u_1}{b_1}\right) - i\frac{a_1}{b_1}x_1\right\}\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}\, e^{\frac{i}{2}\left(\frac{a_j}{b_j}x_j^2 - \frac{2}{b_j}x_ju_j + \frac{d_j}{b_j}u_j^2\right)}$$

$$= \left(-i\frac{u_1}{b_1}\right)\prod_{j=1}^{2}\sqrt{\frac{1}{2i\pi b_j}}\, e^{\frac{i}{2}\left(\frac{a_j}{b_j}x_j^2 - \frac{2}{b_j}x_ju_j + \frac{d_j}{b_j}u_j^2\right)}$$

$$= \left(-i\frac{u_1}{b_1}\right)\prod_{j=1}^{2} K_{A_j}(x_j, u_j)$$

Therefore, Proposition (i) holds for $n = 1$.

Assume that the equation

$$\Lambda_{x_1}^{n-1}\left\{\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\right\} = \left(-i\frac{u_1}{b_1}\right)^{n-1}\prod_{j=1}^{2} K_{A_j}(x_j, u_j)$$

holds. Then,

$$\Lambda_{x_1}^{n}\left\{\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\right\} = \Lambda_{x_1}^{n-1}\left\{\Lambda_{x_1}\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\right\}$$

$$= \Lambda_{x_1}^{n-1}\left\{\left(-i\frac{u_1}{b_1}\right)\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\right\}$$

$$= \left(-i\frac{u_1}{b_1}\right)\Lambda_{x_1}^{n-1}\left\{\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\right\}$$

$$= \left(-i\frac{u_1}{b_1}\right)\left(-i\frac{u_1}{b_1}\right)^{n-1}\prod_{j=1}^{2} K_{A_j}(x_j, u_j)$$

$$= \left(-i\frac{u_1}{b_1}\right)^{n}\prod_{j=1}^{2} K_{A_j}(x_j, u_j)$$

This shows that, by mathematical induction, Proposition (i) holds for all $n \in N$.

The proof of (ii) is similar to (i).

(iii)

$$L_{A_1,A_2}\{\overline{\Lambda}_{x_1}\phi(x_1, x_2)\}(u_1, u_2)$$

$$= \int_{R^2}\left(-\frac{\partial}{\partial x_1} - i\frac{a_1}{b_1}x_1\right)\phi(x_1, x_2)\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\,dx_1dx_2$$

$$= -\int_{R^2}\left\{\frac{\partial}{\partial x_1}\phi(x_1, x_2)\right\}\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\,dx_1dx_2 -$$

$$\int_{R^2}\left(i\frac{a_1}{b_1}x_1\right)\phi(x_1, x_2)\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\,dx_1dx_2$$

$$= \int_{R^2}\phi(x_1, x_2)\frac{\partial}{\partial x_1}\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\,dx_1dx_2 -$$

$$\int_{R^2}\left(i\frac{a_1}{b_1}x_1\right)\phi(x_1, x_2)\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\,dx_1dx_2$$

$$= \int_{R^2}\left(i\frac{a_1}{b_1}x_1 - i\frac{u_1}{b_1}\right)\phi(x_1, x_2)\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\,dx_1dx_2$$

$$-\int_{R^2}\left(i\frac{a_1}{b_1}x_1\right)\phi(x_1, x_2)\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\,dx_1dx_2$$

$$= \left(-i\frac{u_1}{b_1}\right)\int_{R^2}\phi(x_1, x_2)\prod_{j=1}^{2} K_{A_j}(x_j, u_j)\,dx_1dx_2$$

$$= \left(-i\frac{u_1}{b_1}\right)L_{A_1,A_2}\{\phi(x_1, x_2)\}(u_1, u_2)$$

Therefore, Proposition (iii) holds for $n = 1$. Assume that the equation

$$L_{A_1,A_2}\left\{\overline{\Lambda}_{x_1}^{n-1}\phi(x_1,x_2)\right\}(u_1,u_2) = \left(-i\frac{u_1}{b_1}\right)^{n-1} L_{A_1,A_2}\{\phi(x_1,x_2)\}(u_1,u_2)$$

holds. Then

$$L_{A_1,A_2}\left\{\overline{\Lambda}_{x_1}^{n}\phi(x_1,x_2)\right\}(u_1,u_2) = L_{A_1,A_2}\left\{\overline{\Lambda}_{x_1}\left(\overline{\Lambda}_{x_1}^{n-1}\phi(x_1,x_2)\right)\right\}(u_1,u_2)$$

$$= L_{A_1,A_2}\left\{\overline{\Lambda}_{x_1}\left(\overline{\Lambda}_{x_1}^{n-1}\phi(x_1,x_2)\right)\right\}(u_1,u_2)$$

$$= \left(-i\frac{u_1}{b_1}\right) L_{A_1,A_2}\left\{\left(\overline{\Lambda}_{x_1}^{n-1}\phi(x_1,x_2)\right)\right\}(u_1,u_2)$$

$$= \left(-i\frac{u_1}{b_1}\right)\left(-i\frac{u_1}{b_1}\right)^{n-1} L_{A_1,A_2}\{\phi(x_1,x_2)\}(u_1,u_2)$$

$$= \left(-i\frac{u_1}{b_1}\right)^{n} L_{A_1,A_2}\{\phi(x_1,x_2)\}(u_1,u_2)$$

Based on the mathematical induction, the proof of Proposition (iii) is completed.

The proof of proposition (iv) can proved with similar arguments.

5. Applications

In this section, we investigate the application of the two-dimensional Linear Canonical Transform (2D-LCT) to solve the Cauchy problem for the two-dimensional wave equation. Our goal is to derive a solution within the 2D-LCT framework, offering a novel perspective on this classical problem. Previous studies have employed the LCT and FrFT to solve wave, heat and Laplace equations. However, our approach introduces a different methodology specifically tailored to the 2D case, leveraging the properties of the 2D-LCT. To illustrate the effectiveness of this approach, we include a simple example demonstrating the solution process.

Consider the following two dimensional initial value problem for the wave equation associated with the 2D-LCT:

$$\frac{\partial^2\phi(x_1,x_2,t)}{\partial t^2} = c^2\left(\overline{\Lambda}_{x_1}^{2}\phi(x_1,x_2,t) + \overline{\Lambda}_{x_2}^{2}\phi(x_1,x_2,t)\right), -\infty < x_1,x_2 < \infty, t > 0, \quad (3.1)$$

with the initial data

$$\phi(x_1,x_2,0) = f(x_1,x_2), \frac{\partial}{\partial t}\phi(x_1,x_2,0) = g(x_1,x_2), \quad -\infty < x_1,x_2 < \infty, \qquad (3.2)$$

where c is a constant and $\overline{\Lambda}_{x_j} = -\frac{\partial}{\partial x_j} - i\frac{a_j}{b_j}x_j, j = 1,2.$

We apply the 2D-LCT on both sides of (3.1) and using Proposition , we obtain

$$\frac{\partial^2\Phi_{A_1,A_2}(u_1,u_2,t)}{\partial t^2} = c^2\left\{\left(-i\frac{u_1}{b_1}\right)^2 + \left(-i\frac{u_2}{b_2}\right)^2\right\}\Phi_{A_1,A_2}(u_1,u_2,t) = 0$$

which can be written as

$$\frac{\partial^2 \Phi_{A_1,A_2}(u_1,u_2,t)}{\partial t^2} + c^2 \left(\frac{u}{b}\right)^2 \Phi_{A_1,A_2}(u_1,u_2,t) = 0, \tag{3.3}$$

where $\Phi_{A_1,A_2}(u_1,u_2,t) = L_{A_1,A_2}\{\phi(x_1,x_2,t)\}$ is the 2D-LCT of $\phi(x_1,x_2,t)$ and

$$\left(\frac{u}{b}\right)^2 = \left(\frac{u_1}{b_1}\right)^2 + \left(\frac{u_2}{b_2}\right)^2$$

The solution of (3.3) takes the form

$$\Phi_{A_1,A_2}(u_1,u_2,t) = A(u_1,u_2)e^{i\frac{cu}{b}t} + B(u_1,u_2)e^{-i\frac{cu}{b}t} \tag{3.4}$$

Differentiating (3.4) w.r.t. t, we get

$$\frac{\partial}{\partial t}\Phi_{A_1,A_2}(u_1,u_2,t) = i\frac{cu}{b}A(u_1,u_2)e^{i\frac{cu}{b}t} - i\frac{cu}{b}B(u_1,u_2)e^{-i\frac{cu}{b}t} \tag{3.5}$$

Using initial data in (3.4) and (3.5), we obtain

$$\mathrm{F}_{A_1,A_2}(u_1,u_2) = \Phi_{A_1,A_2}(u_1,u_2,0) = L_{A_1,A_2}\{\phi(x_1,x_2,0)\} = L_{A_1,A_2}\{f(x_1,x_2)\}$$

$$= A(u_1,u_2) + B(u_1,u_2) \tag{3.6}$$

$$\mathrm{G}_{A_1,A_2}(u_1,u_2) = \frac{\partial}{\partial t}\Phi_{A_1,A_2}(u_1,u_2,0) = L_{A_1,A_2}\left\{\frac{\partial}{\partial t}\phi(x_1,x_2,0)\right\} = L_{A_1,A_2}\{g(x_1,x_2)\}$$

$$= i\frac{cu}{b}\{A(u_1,u_2) - B(u_1,u_2)\} \tag{3.7}$$

Solving (3.6) and (3.7) for $A(u_1,u_2)$ and $B(u_1,u_2)$, we obtain

$$A(u_1,u_2) = \frac{1}{2}\left\{\mathrm{F}_{A_1,A_2}(u_1,u_2) + \frac{b}{icu}\mathrm{G}_{A_1,A_2}(u_1,u_2)\right\} \text{ and} \tag{3.8}$$

$$B(u_1,u_2) = \frac{1}{2}\left\{\mathrm{F}_{A_1,A_2}(u_1,u_2) - \frac{b}{icu}\mathrm{G}_{A_1,A_2}(u_1,u_2)\right\} \tag{3.9}$$

Substituting (3.8) and (3.9) in (3.4) yields

$$\Phi_{A_1,A_2}(u_1,u_2,t)$$

$$= \frac{1}{2}\left\{\mathrm{F}_{A_1,A_2}(u_1,u_2) + \frac{b}{icu}\mathrm{G}_{A_1,A_2}(u_1,u_2)\right\}e^{i\frac{cu}{b}t} + \frac{1}{2}\left\{\mathrm{F}_{A_1,A_2}(u_1,u_2) - \frac{b}{icu}\mathrm{G}_{A_1,A_2}(u_1,u_2)\right\}e^{-i\frac{cu}{b}t}$$

$$= \mathrm{F}_{A_1,A_2}(u_1,u_2)\left\{\frac{e^{i\frac{cu}{b}t} + e^{-i\frac{cu}{b}t}}{2}\right\} + \frac{b}{cu}\mathrm{G}_{A_1,A_2}(u_1,u_2)\left\{\frac{e^{i\frac{cu}{b}t} - e^{-i\frac{cu}{b}t}}{2i}\right\}$$

$$= \mathrm{F}_{A_1,A_2}(u_1,u_2)\cos\left(\frac{cu}{b}t\right) + \frac{b}{cu}\mathrm{G}_{A_1,A_2}(u_1,u_2)\sin\left(\frac{cu}{b}t\right) \tag{3.10}$$

Applying inverse of 2D-LCT on equation (3.10), we obtain

$$\phi(x_1, x_2, t)$$

$$= \int_{R^2} \mathrm{F}_{A_1,A_2}(u_1,u_2)\cos\left(\frac{cu}{b}t\right)\prod_{j=1}^{2}\sqrt{\frac{1}{-2i\pi b_j}}\,e^{-\frac{i}{2}\left(\frac{a_j}{b_j}x_j^2-\frac{2}{b_j}x_ju_j+\frac{d_j}{b_j}u_j^2\right)}du_1du_2$$

$$+\int_{R^2}\frac{b}{cu}\mathrm{G}_{A_1,A_2}(u_1,u_2)\sin\left(\frac{cu}{b}t\right)\prod_{j=1}^{2}\sqrt{\frac{1}{-2i\pi b_j}}\,e^{-\frac{i}{2}\left(\frac{a_j}{b_j}x_j^2-\frac{2}{b_j}x_ju_j+\frac{d_j}{b_j}u_j^2\right)}du_1du_2$$

6. Conclusion

This work established key operational properties of the two-dimensional Linear Canonical Transform, including linearity, shifting, scaling, modulation, and axis inversion. These results strengthen the theoretical framework of the 2D-LCT and support its use in multidimensional signal analysis. The successful application of the transform to the two-dimensional wave equation demonstrates its effectiveness in solving partial differential equations. Overall, the findings highlight the versatility and analytical value of the 2D-LCT for future research in mathematical physics and signal processing.

References

1. Healy, John J., et al. "Linear canonical transforms." *Springer Series in Optical Sciences* 198 (2016): 453.
2. Moshinsky, Marcos, and Christiane Quesne. "Linear canonical transformations and their unitary representations." *Journal of Mathematical Physics* 12.8 (1971): 1772-1780.
3. Li, Bing-Zhao, Ran Tao, and Yue Wang. "New sampling formulae related to linear canonical transform." *Signal Processing* 87.5 (2007): 983-990.
4. Zhao, Weikang, et al. "Research on Two-Dimensional Linear Canonical Transformation Series and Its Applications." *Fractal and Fractional* 9.9 (2025): 596.
5. Deng, Bing, Ran Tao, and Yue Wang. "Convolution theorems for the linear canonical transform and their applications." *Science in China Series F: Information Sciences* 49.5 (2006): 592-603.
6. Wei, Deyun, et al. "A convolution and product theorem for the linear canonical transform." *IEEE Signal Processing Letters* 16.10 (2009): 853-856.
7. Wei, Deyun, Qiwen Ran, and Yuanmin Li. "A convolution and correlation theorem for the linear canonical transform and its application." *Circuits, Systems, and Signal Processing* 31.1 (2012): 301-312.
8. Zhang, Zhi-Chao. "Linear canonical transform's differentiation properties and their application in solving generalized differential equations." *Optik* 188 (2019): 287-293.
9. Yang, Yinuo, Qingyan Wu, and Seong-Tae Jhang. "2D linear canonical transforms on L P and applications." *Fractal and Fractional* 7.2 (2023): 100.