



AN ALGORITHMIC STUDY OF THE STATIONARY CHARACTERISTICS OF AN (S-1, S) PRODUCTION-INVENTORY SYSTEM WITH PHASE TYPE DISTRIBUTION

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Abstract : *A continuous review production-inventory system with (S-1, S) policy, which means that as soon as we sell one, we start production of another, is considered in this paper. This policy is typically used for long production time, capital intensive, low demand service parts in industries such as aerospace. The demand is according to a Markovian Arrival Process (MAP) with inter-arrival times following a continuous phase type (CPH) distribution. The production times are also assumed to follow a CPH distribution. Phase Type(PH) distribution has gained widespread acceptance because of its versatility and computational properties in applied stochastic modelling. We propose an algorithmic solution of this production-inventory system for its stationary characteristic and implement it using 'R' code. Within this framework a number of model variants are solved.*

Keywords: Production inventory model, (S-1,S) policy, Phase type distribution, Markovian arrival process

INTRODUCTION

The management of all the available resources is the main issue facing by almost all companies. These resources include labour, equipment, money, land, information etc. Most of the time this resource management techniques they are adopting help them to capture the market. Mainly different mathematical models have been used in providing solutions to manage these resources. Here in this context, the term inventory becomes a necessity for the industry field. It ensures a continuous supply of products to the customers.

The widespread use of inventory control in the industrial field made an increasing interest in the study of inventory management. Studies that began in the 20th century have become increasingly paramount after the World War II. Thereafter a number of important literature have emerged which includes inventory modelling, its analysis, and its optimization. The study on inventory control deals with two types of problems such as single-item and multi item problems. The mathematical inventory models can be divided into two broad

categories – deterministic models and stochastic models, according to the predictability of input parameters involved. Inventory systems are also classified as continuous–review systems and periodic review systems.

A number of possible inventory control policies are introduced for inventory systems. Mainly the inventory policy is concerned with two decisions: when to reorder and how much to reorder. Much of the complexity of inventory analysis springs from the interaction of these two decisions. A special but important inventory problem where the items whose cost is so high, demand is so low that the optimal policy is to place a reorder whenever a demand occurs, for example, most aircraft recoverable spare parts that may cost hundreds of dollars and are typically demanded only a few times a year at an individual air base, can be considered as a special case of the important (s, S) policy. The policy is called the $(S-1, S)$ policy. Here, this policy in the continuous review environment will place an order immediately whenever a demand occurs for one or more units of an item. The order quantity matches the size of the demand exactly. Thus, the inventory position is constant in the case where the demand process and costs are stationary over an infinite planning horizon. The major objective while using this policy is to show how to compute the stationary probability distribution of the quantity of units in resupply. The amount in resupply at a random point in time is a key random variable in the study of the behavior of systems managed using an $(S-1, S)$ policy. Once its stationary distribution is known, we can easily determine the stationary distribution for on-hand and back-ordered inventory.

There are several basic considerations involved in determining an inventory policy that must be reflected in the mathematical inventory model. That is various costs and different variables such as control variables and non-control variables are incorporated, which may be deterministic or stochastic in nature, completely predictable, or predictable up to some probability distribution only, probability distribution may even be unknown. In a production inventory models the major consideration is given to the production line and demand during a specific period. We can make use of different distribution to model production times and demand of this kind of models. One such important distribution we can make use in this scenario is the phase type (PH) distribution. Here in this work we use a kind of $(S-1, S)$ inventory model which uses the PH distribution.

1.1 Literature Review

A detailed discussion of $(S-1, S)$ policy is given in (Hadley and Whitten, 1963) and (Sherbrooke, 1968). (Higa et al., 1975) studied the customer waiting in an $(S-1, S)$ production/ inventory system with geometric Poisson demand arrivals and exponentially distributed replacement times. (Smith, 1977) obtained an appropriate formula for minimizing the cost with respect to S in an $(S-1, S)$ pure inventory system with Poisson demand arrivals and an arbitrary replenishment time distribution. (Muckstadt and Thomas, 1980), (Shanker, 1981), (Hausman and Scudder, 1982), (Graves, 1985) and (Sherbrooke, 1968) are some of the works on of multi-echelon repairable systems employing the $(S-1, S)$ policy.

A number of works that are discussed in literature on uses of phase type distribution in inventory modeling. In all the works, phase type was used to represent uncertain demand or uncertain leadtime distribution or as both. Since in some cases certain inventory systems consider as absorbing Markovian process and can make use of phase type distribution there. (Zipkin, 1988) showed that the marginal distribution of leadtime demand has a discrete phase type distribution with the same number of phases as the leadtime distribution. Also the use of phase type is more effective in high cost items which need more careful attention in modeling and analysis. (Chakravarthy, 2010) considered a (s, S) – type inventory model with demands are arrived according to a class of point process having phase type holding and replenishment times. (Radhamani et al., 2014) used discrete phase type as a leadtime distribution to model a discrete-time inventory system in which demand is a discrete time Markovian arrival process. In 2016, (Barron et al., 2016) published a work on Markovian additive production-inventory process model with phase type demand arrival.

II. PHASE TYPE DISTRIBUTION

Phase type(PH) distributions are a versatile tool for modelling a wide range of real-world phenomena. The method of phases (or stages) may be thought of as an approximation procedure according to which a random variable with an arbitrary distribution is represented by either a sum or a mixture or by a combined sum-mixture of independent (not necessarily identical) random variables, each being a phase or a stage of the ‘life time’ of the original random variable. Though PH distributions can be traced back to the pioneering work of (Erlang, 1917) and to (Jensen, 1954), it was not until the late seventies that (Neuts, 1981), (Neuts, 1989), (Neuts, 1995) and co-workers established much of the modern theory available today. Phase type distribution is the distribution of the random variable describing the time until reaching the absorbing state of a Markov chain with m transient states and one absorbing state. The underlying Markov chain is called a representation of the distribution. Each of the states of the Markov chain represents one of the phases. Phase type distributions can be defined based on a discrete time Markov chain leading to a discrete phase type(DPH) on \mathbb{N} as well as in the continuous case leading to a continuous phase type(CPH) distribution on \mathbb{R}^+ .

Now let us briefly discuss the DPH distribution and CPH distribution.

2.1 Discrete Phase Type Distribution

A discrete phase type distribution (DPH) is the distribution of the time to absorption in a finite discrete time Markov chain with transition matrix \mathbf{P} of dimension $m + 1$ is given by (1). The Markov chain has m transient and one absorbing state.

$$\mathbf{P} = \begin{pmatrix} \mathbf{T} & \mathbf{T}^0 \\ \mathbf{0} & 1 \end{pmatrix}, \quad (1)$$

where

$$\begin{aligned} \mathbf{T} &= (T_{ij})_{m \times m}; \\ \mathbf{T}^0 &= (T_1^0, T_2^0, T_3^0, \dots, T_m^0)'; \\ \mathbf{0} &= (0, 0, \dots, 0)_{1 \times m} \end{aligned}$$

and

$$\mathbf{T}\mathbf{e} + \mathbf{T}^0 = \mathbf{e} = (1, 1, \dots, 1)'$$

Here $(\mathbf{I} - \mathbf{T})$ is non-singular, that is, the states $1, 2, 3, \dots, m$ are transient and the absorption into the state $(m + 1)$ starting from each of them is certain. The initial probability vector of the Markov chain $\{Y_n, n \geq 0\}$ with tpm \mathbf{P} is denoted by $\underline{\alpha}^* = (\underline{\alpha}, \alpha_{m+1})$, where $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$. The pair $(\underline{\alpha}, \mathbf{T})$ is called a representation for the PH distribution. That is, if we take τ as the time until absorption then,

$$\tau = \inf\{n \geq 0 \mid Y_n = m + 1\} \text{ and } \tau \sim PH(\underline{\alpha}, \mathbf{T})$$

Definition 2.1 A probability distribution $\{p_k\}$ on the nonnegative integers is of phase type if there exists \mathbf{P} and α^* of the type defined above such that $\{p_k\}$ is the probability that at step k , the Markov Chain $\{Y_n, n \geq 0\}$ with tpm \mathbf{P} and initial probability vector α^* gets absorbed into the state $(m + 1)$. The probability distribution $\{p_k, k \geq 0\}$ in this case will be called discrete phase type distribution(DPH).

Example 2.1 Geometric Distribution $p_k = pq^k, k \geq 0, 0 < p < 1, p + q = 1$ has a DPH (α, \mathbf{T}) of order 1, where $\underline{\alpha} = q$ and $\mathbf{T} = q$.

2.2 Continuous Phase Type Distribution

Continuous phase type distribution (CPH) is obtained as the distribution of the time until absorption in a continuous time Markov chain. Thus the CPH is the continuous analogue to DPH.

Now let us consider an absorbing continuous Markov chain, $\{Y_t, t \geq 0\}$ with state space $\omega = \{1, 2, \dots, m, m+1\}$. Let the infinitesimal generator of this chain be

$$Q = \begin{pmatrix} \mathbf{T} & \mathbf{T}^0 \\ \mathbf{0} & 0 \end{pmatrix} \quad (2)$$

where $\mathbf{T} = (T_{ij})_{m \times m}$ satisfies $T_{ii} < 0$, for $1 \leq i \leq m$ and $T_{ij} \geq 0$ for $i \neq j, 1 \leq i, j \leq m$; $\mathbf{T}^0 = (T_1^0, T_2^0, T_3^0, \dots, T_m^0)'_{m \times 1}$ and $\mathbf{T}\mathbf{e} + \mathbf{T}^0 = \mathbf{0}$. The initial probability vector is given by, $\underline{\alpha}^* = (\underline{\alpha}, \alpha_{m+1})$ such that $\underline{\alpha}\mathbf{e} + \alpha_{m+1} = 1$.

Then the following results will hold,

- i. P_i^* (the probability of eventual absorption into the state $m+1$, starting from state i (for $1 \leq i \leq m$), satisfy the system of linear equations $\mathbf{TP}^* = -\mathbf{T}^0$ where $\mathbf{P}^* = (P_i^*)_{m \times 1}$.
- ii. States $1, 2, \dots, m$ are transient if and only if the matrix \mathbf{T} is nonsingular.
- iii. The cumulative probability distribution $F(\cdot)$ of the time until absorption into state $m+1$, is given by, $F(x) = 1 - \underline{\alpha} \exp(\mathbf{T}x) \mathbf{e}$ for $x \geq 0$.

Definition 2.2 A probability distribution $F(\cdot)$ on $[0, \infty)$ is said to be continuous phase type (CPH) distribution if it is the distribution of the time until absorption in an absorbing continuous parameter Markov chain of the type defined above. The pair $(\underline{\alpha}, \mathbf{T})_m$ is called a representation of $F(\cdot)$ and m is called the order.

Example 2.2 A simple example of CPH is the exponential distribution $(F(x) = 1 - e^{-\lambda x}, x \geq 0, \lambda > 0)$, which has the CPH representation $(\underline{\alpha}, \mathbf{T})_m$ with $\alpha = 1, \mathbf{T} = -\lambda$ and $m = 1$.

III. MODEL SPECIFICATION

3.1 Assumptions and Variables

- Time between successive demands is assumed to follow a PH distribution $F(\cdot)$ which has an irreducible form $(\underline{\alpha}, \mathbf{T})_m$ of order m .
- Production times are also phase type distributed, $G(\cdot)$ with the irreducible form $(\underline{\beta}, \mathbf{U})_n$ of order n .
- S is the maximum possible stock on hand.
- π_ω represents the steady state probability of the state $\omega \in \Omega$.
- p_0 and p_i denotes the stationary on-hand probabilities.

3.2 Stationary Distribution of the Proposed Model

(Jenson, 2001), considered a production-inventory $(S-1, S)$ model in which there is a production facility and a finished product warehouse for a single commodity. The production continues until the stock on hand reaches S . If all the S spaces are fill, then the production stops, and in a similar manner no demand will be entertained if all the S spaces are vacant (demand will be rejected at that time). In this model time between successive demands is as summed to follow a PH distribution $F(\cdot)$ which has an irreducible form $(\underline{\alpha}, \mathbf{T})_m$ of order m . Here the production times are independent of the demand process and are mutually independent having distribution of phase type, $G(\cdot)$ with the irreducible form $(\underline{\beta}, \mathbf{U})_n$ of order n . That is, \mathbf{T} and \mathbf{U} are the generators of Markov process describing the generation of demand and production. Now consider stock level as a stochastic process $\{I(t), t \geq 0\}$ with $I(0) = S$. Thus $I(t)$ denotes the on-hand inventory level at time t and its probability mass function is the study of interest, that is $P\{I(t) = n\}, n = 0, 1, 2, \dots, S$.

Now let us denote $X(t)$ as the state of the inventory at time t . Then $\{X(t), t \geq 0\}$ may be considered as homogeneous Markov process on the state space $\Omega = \{(0, j); j = 1, 2, \dots, m\} \cup \{(i, j, k) / i = 1, 2, \dots, S; j = 1, 2, \dots, m; k = 1, 2, \dots, n\}$, where $(0, j)$ represents the system is empty and the demand is at phase j , while (i, j, k) represents that there are i units in the system and is in the j^{th} demand phase and k^{th} production phase.

Using the properties and procedures described in (Neuts, 1981), (Jenson, 2001) derived an algorithm to find the steady state probability of the state $\omega \in \Omega$, generally denoted as $\underline{\pi}_\omega$ and defined by,

$$\underline{\pi}_0 = (\pi_{01}, \pi_{02}, \dots, \pi_{0m}), \quad \text{and}$$

$$\underline{\pi}_i = (\pi_{i,1,1}, \pi_{i,1,2}, \dots, \pi_{i,1,m}, \pi_{i,2,1}, \dots, \pi_{i,m,n}); i = 1, 2, \dots, S.$$

These $\underline{\pi}_\omega$ values are then used to obtain the stationary on-hand probabilities denoted and defined as

$$p_0 = \underline{\pi}_0 \underline{\mathbf{e}}_m \quad \text{and} \quad p_i = \underline{\pi}_i \underline{\mathbf{e}}_{mn}; i = 1, 2, \dots, S.$$

where $\underline{\mathbf{e}}_m$, $\underline{\mathbf{e}}_n$ and $\underline{\mathbf{e}}_{mn}$ are column vectors of order m , n and mn , respectively, with their all components equals to one.

IV. ALGORITHM FOR THE CALCULATION OF STATIONARY PROBABILITY DISTRIBUTION OF INVENTORY LEVEL

By the analysis of (Jenson, 2001) the following algorithm was suggested for the computation of stationary probabilities $\{p_i; i = 1, 2, \dots, S\}$

Inputs:

Integers m, n , and S
 Row vectors $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$
 $\underline{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$
 Matrices $\mathbf{T} = (T_{ij})_{m \times m}$
 $\mathbf{U} = (U_{ij})_{n \times n}$

Calculations:

$$\underline{\mathbf{T}}^0 = -\mathbf{T} \underline{\mathbf{e}}_m$$

$$\underline{\mathbf{U}}^0 = -\mathbf{U} \underline{\mathbf{e}}_n$$

$$\mathbf{U}^* = \left(\mathbf{T} \otimes (\underline{\mathbf{e}}_n \underline{\beta} - \mathbf{I}_n) \right) - (\mathbf{I}_m \otimes \mathbf{U})$$

$$\mathbf{T}^* = \left((\underline{\mathbf{e}}_m \underline{\alpha} - \mathbf{I}_m) \otimes \mathbf{U} \right) - (\mathbf{T} \otimes \mathbf{I}_n)$$

$$\mathbf{M}_0 = -(\mathbf{T} \otimes \underline{\beta}) \mathbf{U}^{*-1}$$

$$\mathbf{M} = \mathbf{T}^* \mathbf{U}^{*-1}$$

$$\mathbf{M}_{S-1} = -(\underline{\mathbf{T}}^0 \underline{\alpha} \otimes \mathbf{I}_n) \left[(\mathbf{T} + \underline{\mathbf{T}}^0 \underline{\alpha}) \otimes \mathbf{I}_n + (\mathbf{I}_m \otimes \mathbf{U}) \right]^{-1}$$

$$\mathbf{W} = \mathbf{I}_m + \left((\mathbf{M}_0 \mathbf{M}^{S-2} \mathbf{M}_{S-1}) + \sum_{i=0}^{S-2} (\mathbf{M}_0 \mathbf{M}^i) \right) (\mathbf{I}_n \otimes \underline{\mathbf{e}}_n)$$

Now solve for $\underline{v} = (v_1, v_2, \dots, v_m)$ from the system of equations

$$\underline{v}(\mathbf{T} + \mathbf{T}^0\alpha) = \mathbf{0} \text{ and}$$

$$\underline{v} \mathbf{e}_m = \mathbf{1}$$

and then solve $\underline{\pi}_0 = (\pi_{01}, \pi_{02}, \dots, \pi_{0m})$, from $\underline{\pi}_0 \mathbf{W} = \underline{v}$. Thus we will get the desired steady state distribution functions as

$$\underline{\pi}_i = \begin{cases} \underline{\pi}_0 \mathbf{M}_0 \mathbf{M}^{i-1} & \text{for } i = 1, 2, \dots, S-1 \\ \underline{\pi}_0 \mathbf{M}_0 \mathbf{M}^{S-2} \mathbf{M}_{S-1} & \text{for } i = S. \end{cases}$$

The Stationary Probabilities:

$$p_i = \underline{\pi}_i \mathbf{e}_{mn}; i = 1, 2, \dots, S.$$

$$p_0 = \underline{\pi}_0 \mathbf{e}_m$$

V. R-CODE FOR THE ALGORITHM OF (S-1,S) PRODUCTION-INVENTORY SYSTEM

As *R* is an efficient and widely accepted statistical software, it is quite helpful if we derive an *R* code for the above discussed algorithm. An attempt is made here to obtain an *R*-code to calculate the stationary probabilities of this stochastic process. Thus *R* programme for finding the stationary distribution of this system based on the previous algorithm and a numerical illustration and its graph of the stationary probabilities given below.

Using the pre-existing **expm** and **Matrix** packages for optimization and matrix operations we have derived a function to obtain the stationary probabilities of the production inventory model.

stat_dist = function(m, n, S, alpha, beta, T, U)

Here, the stationary distribution is obtained by giving the initial probability vectors (alpha, beta) and infinitesimal generators (T, U) of the production and demand line in the **stat_dist** function. *m* and *n* are the dimensions of **T** and **U**, respectively. Within this framework a number of model variants are solved, such as exponential, Erlang, mixture of Erlang, Coxian models etc. Next section deals with these illustrations.

VI. ILLUSTRATIONS

As we discussed in the previous section most of the life distributions can be represented as phase type distribution. The following subsections include the stationary behaviour of different production-inventory system having some life distributions as the demand and production time distributions.

6.1 Exponential

Suppose that in a production-inventory system with $S = 5$ the inter-demand times follows a CPH $(\underline{\alpha}, \mathbf{T})_m$ where $m = 3, \underline{\alpha} = 1$ and $\mathbf{T} = -1$. The production times follows a CPH $(\underline{\beta}, \mathbf{U})_n$ where $n = 3, \underline{\beta} = 1$ and $\mathbf{U} = -1$. That is, the demand and production times are both follow exponential distributions. Then the stationary probabilities of this production inventory model can be obtained by calling the above mentioned **stat_dist** function. The *R* code for this model and the corresponding plot(Figure 1) is as follows;

#Exponential Distribution

m = 1

n = 1

$S = 5$

$\alpha = \text{matrix}(1,1,m)$

$\beta = \text{matrix}(1,1,n)$

$T = \text{matrix}(-1,m,m,T)$

$U = \text{matrix}(-1,n,n,T)$

$\text{stat_dist}(m, n, S, \alpha, \beta, T, U)$

Graph of Stationary Distribution

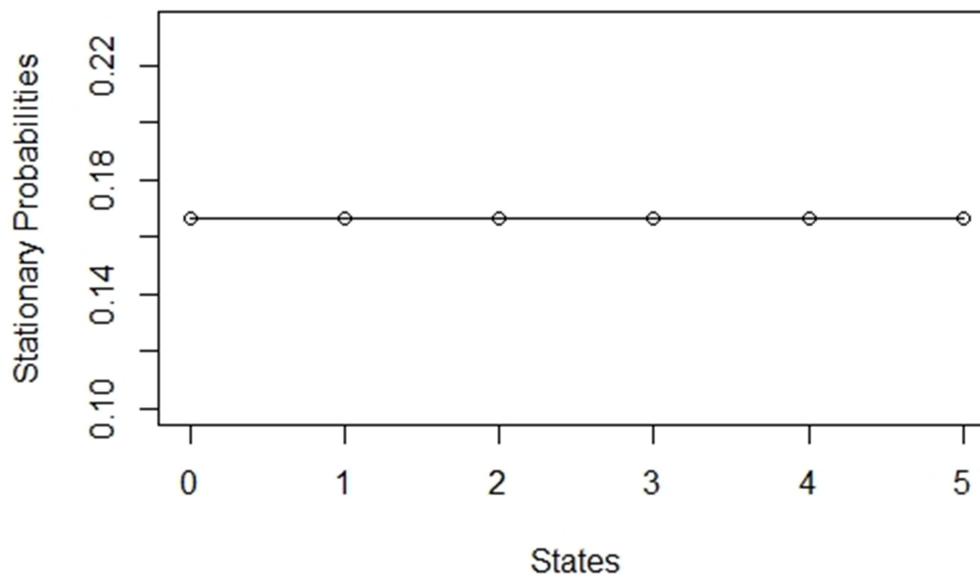


Figure 1: Exponential Distribution

6.2 Erlang Distribution

Now consider another production-inventory system with $S = 14$ the inter demand times follows a CPH

$(\underline{\alpha}, \mathbf{T})_m$ where $m = 3, \underline{\alpha} = (1, 0, 0)$ and $\mathbf{T} = \begin{pmatrix} -2.5 & 2.5 & 0 \\ 0 & -2.5 & 2.5 \\ 0 & 0 & -2.5 \end{pmatrix}$. The production times follows a CPH

$(\underline{\beta}, \mathbf{U})_n$ where $n = 4, \underline{\beta} = (1, 0, 0, 0)$ and $\mathbf{U} = \begin{pmatrix} -3.5 & 3.5 & 0 & 0 \\ 0 & -3.5 & 3.5 & 0 \\ 0 & 0 & -3.5 & 3.5 \\ 0 & 0 & 0 & -3.5 \end{pmatrix}$. Here the inter-demand and

production times follows Erlang distribution. Using a similar R code that is used in the previous subsection, we can obtain the stationary distribution of this production-inventory system. The graph showing the stationary distribution of this system is given in Figure 2.

Graph of Stationary Distribution

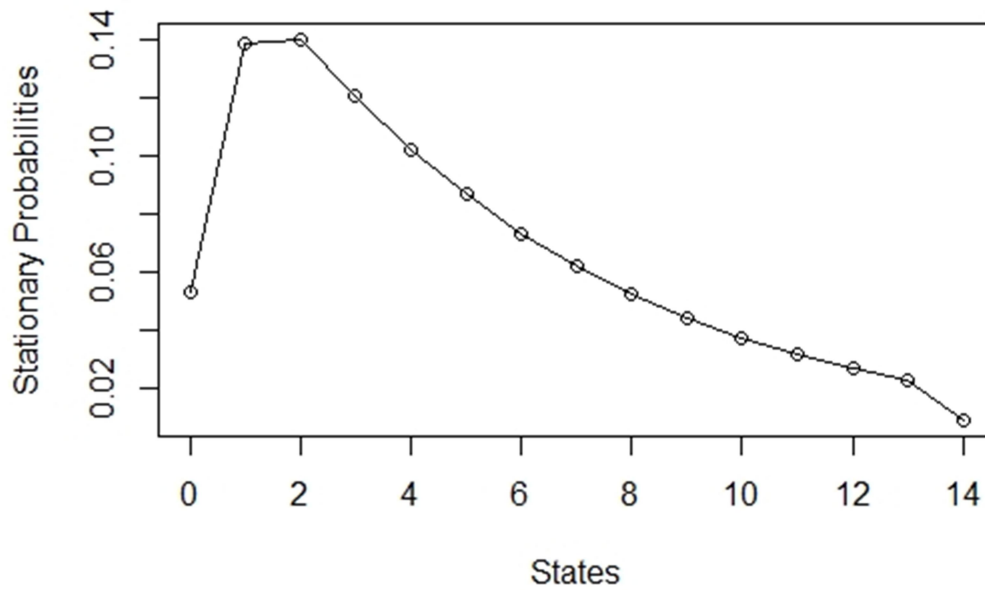


Figure 2: Erlang Distribution

6.3 Mixture of Erlang Distributions

Let us now consider a production-inventory system whose inter-demand and production times, both follow mixture of Erlang distribution. Here $S = 9$, the inter-demand times can be written as a CPH $(\underline{\alpha}, \mathbf{T})_m$, where $m = 8$, $\underline{\alpha} = (2/3, 0, 0, 0, 1/3, 0, 0, 0)$,

$$\mathbf{T} = \begin{pmatrix} -1.5 & 1.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 1.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 1.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.5 & 3.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.5 & 3.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.5 & 3.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.5 \end{pmatrix}$$

The production times have CPH $(\underline{\beta}, \mathbf{U})_n$, where $n = 8$, $\underline{\beta} = (2/3, 0, 0, 0, 1/3, 0, 0, 0)$,

$$\mathbf{U} = \begin{pmatrix} -1.5 & 1.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 1.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 1.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.5 & 3.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.5 & 3.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.5 & 3.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.5 \end{pmatrix}$$

The R code used in subsection 6.1 for this system gives the stationary distribution graph(see Figure 3).

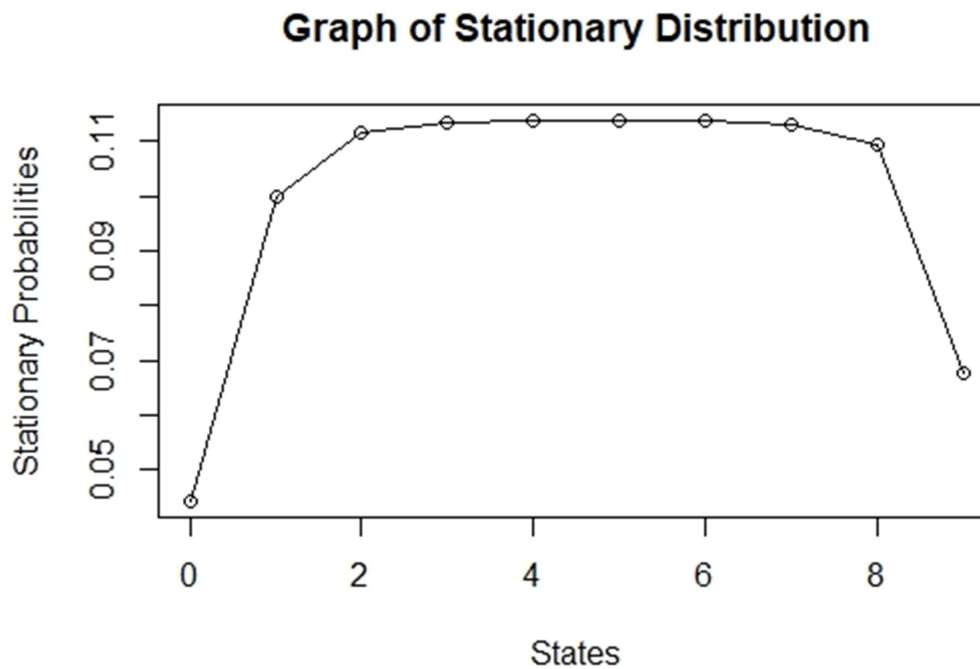


Figure 3: Mixture of Erlang Distribution

6.4 Coxian Distribution

We can use the Coxian distribution as the inter-demand and production time distribution. Consider such a production-inventory system with $S = 9$ the inter-demand times follows a CPH $(\underline{\alpha}, \mathbf{T})_m$, where $m = 3$, $\underline{\alpha} = (1, 0, 0)$ and $\mathbf{T} = \begin{pmatrix} -1 & 1/2 & 0 \\ 0 & -2 & 2/3 \\ 0 & 0 & -3 \end{pmatrix}$. The production time follows a CPH $(\underline{\beta}, \mathbf{U})_n$ where $n = 3$, $\underline{\beta} = (1, 0, 0)$ and $\mathbf{U} = \begin{pmatrix} -1 & 1/3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{pmatrix}$. Graph of the stationary distribution(see Figure 4) of this production-inventory system is obtained using the similar R code that is discussed in subsection 6.1.

Graph of Stationary Distribution

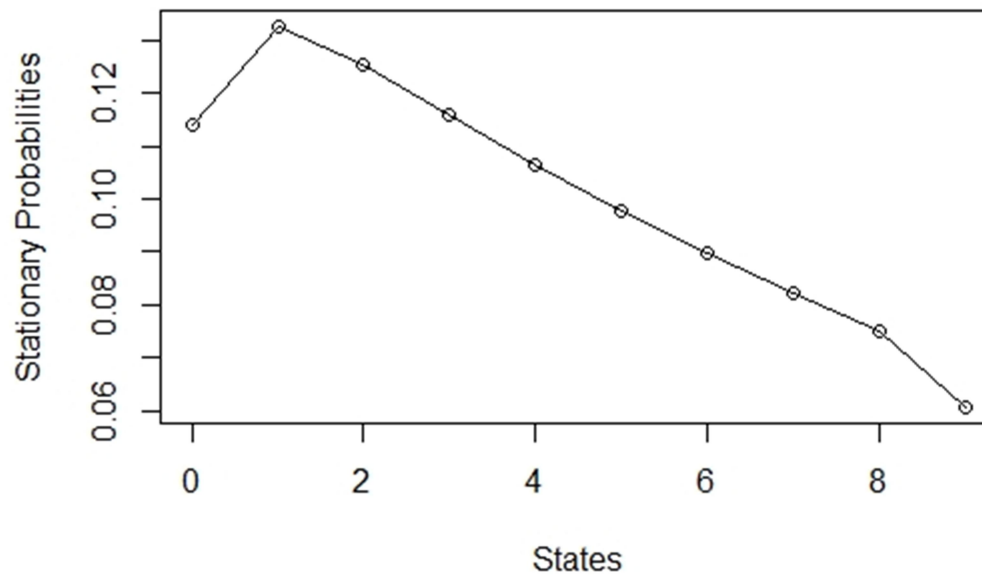


Figure 4: Coxian Distribution

6.5 Erlang inter-demand distribution and Coxian production time distribution System

So far we considered the system having similar inter-demand and production-time distributions. Now consider a production inventory model with $S = 14$ the inter-demand times follows a CPH $(\underline{\alpha}, \mathbf{T})_m$, where $m = 3$,

$\underline{\alpha} = (1, 0, 0)$ and $\mathbf{T} = \begin{pmatrix} -2.5 & 2.5 & 0 \\ 0 & -2.5 & 2.5 \\ 0 & 0 & -2.5 \end{pmatrix}$. The production time follows a CPH $(\underline{\beta}, \mathbf{U})_n$ where $n = 3$,

$\underline{\beta} = (1, 0, 0)$ and $\mathbf{U} = \begin{pmatrix} -1 & 1/3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{pmatrix}$. In the above mentioned system the inter-demand times follow an

Erlang distribution and production-times follow a Coxian distribution. The R code discussed in section 6.1 gives the graph of the stationary distribution of this system(see Figure 5).

Graph of Stationary Distribution

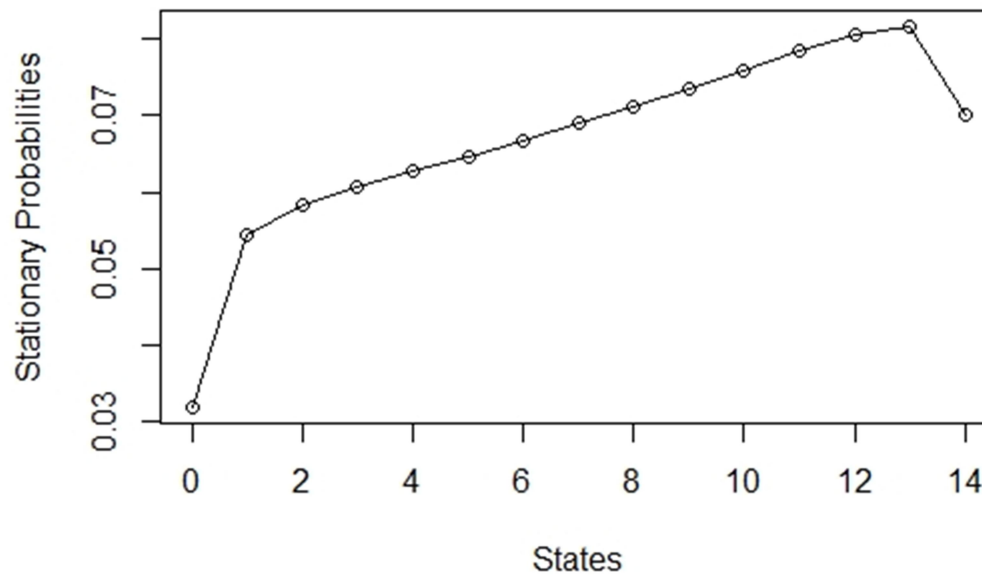


Figure 5: Erlang-Coxian Model

VII. CONCLUSION

The major objective while using $(S - 1, S)$ policy is to show how to compute the stationary probability distribution of the quantity of units in resupply. The amount in resupply at a random point in time is a key random variable in the study of the behavior of systems managed using $(S - 1, S)$ policy. Once its stationary distribution is known, we can easily determine the stationary distribution for on-hand and back-ordered inventory.

In this paper, we considered a $(S - 1, S)$ production-inventory model in which production-times and inter-demand times follow certain continuous phase-type distribution. We developed an algorithm to find the stationary behaviour of this system. As the manual calculation is too complex, we derived *R* code for this system. Now one can impose cost structure on the process and the conditional expected cost per period, given the inventory level at the beginning of the period, averaged with respect to this stationary distribution. This average cost can be minimized with respect to the parameters that characterize the policy being used. In that way the steady-state or stationary solutions to this inventory problem can be obtained.

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