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ON THE INTEGRATION OF PRODUCT OF WHITTAKER FUNCTION WITH RESPECT TO THE GENERALIZED LOMMEL WRIGHT K-FUNCTION

Sapna Chanderiya[1], Dr. Indu Bala Bapna[2]
Vice Principal[1] Principal[2]
[1]*Department of Mathematics, Maharshi Dayanand Saraswati University, Ajmer, Rajasthan, India*
[2] *Department of Mathematics, Maharshi Dayanand Saraswati University, Ajmer, Rajasthan, India*
sapnachanderiya5@gmail.com, *Rajasthan, India*

Abstract : In the present scenario we introduces an integration of product of Whittaker function and the Lommel Wright k- function. these integration are expressed in terms of the wright hypergeometric k-function. various interesting consequence are obtained in this paper. which plays an important role to solve many problems and results related to Whittaker function and generalized Lommel Wright k-function.
Keywords- Generalized Lommel Wright K-function, Whittaker function.

I. INTRODUCTION

In recent years the fractional calculus has become one of the most rapidly growing research subject area of mathematical analysis due to its various application in numerous parts of science along with mathematics.

The Wright Hypergeometric function in series form [3], denoted by ${}_p\psi_q(z)$ defined as:

$$ {}_p\psi_q\left[\begin{matrix}(\alpha_1,A_1)\cdots(\alpha_p,A_p) \\ (\beta_1,B_1)\cdots(\beta_q,B_q)\end{matrix} \quad ; \quad z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+A_i n)z^n}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j n)n!} $$

where the coefficient $A_1 \dots .. A_p$ and $B_1 \dots .. B_q$ are positive real numbers such that.

$$ 1+\sum_{j=1}^{q}B_j - \sum_{i=1}^{p}A_i \geq 0 $$

Gehlot and Prajapati defined the Lommel Wright k function as follows [9].

For $k \in R^+, w, \alpha_i, \beta_j \in C \;\&\; A_i, B_j \in R(A_i, B_j)$ *where* $i = 1,2, \dots .\mathrm{p} \;\&\; j = 1,2, \dots .\mathrm{q}$ *and* $(\alpha_i + nA_i), (\beta_j + nB_j) \in C/kz^-$.

We take the Wright Hypergeometric k-function in series form [8], denoted by ${}_p\psi_q^k$ defined as:

$$ {}_p\psi_q^k\left[\begin{matrix}(\alpha_1,kA_1)\cdots(\alpha_p,kA_p) \\ (\beta_1,kB_1)\cdots(\beta_q,kB_q)\end{matrix} \quad ; \quad z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(\alpha_i+nkA_i)}{\prod_{j=1}^{q}\Gamma_k(\beta_j+nkB_j)}\frac{z^n}{n!} $$

Where the coefficients $A_1 \dots . A_p$ and $B_1 \dots . B_q$ are positive real number such that

$$ 1+\sum_{j=1}^{q}\frac{B_j}{k} - \sum_{i=1}^{p}\frac{A_i}{k} \geq 0 $$

and slightly generalized form is

$$ {}_p\psi_q^k\left[\begin{matrix}(\alpha_1,\mathrm{k})\cdots(\alpha_p,\mathrm{k}) \\ \\ (\beta_1,\mathrm{k})\cdots(\beta_q,\mathrm{k})\end{matrix} \quad ; \quad z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(\alpha_i)}{\prod_{j=1}^{q}\Gamma_k(\beta_j)}\ {}_pF_q^k\left[\begin{matrix}(\alpha_1,\mathrm{k})\cdots(\alpha_p,\mathrm{k}) \\ \\ (\beta_1,\mathrm{k})\cdots(\beta_q,\mathrm{k})\end{matrix} \quad ; \quad z\right] $$

Where ${}_pF_q^k$ is the generalized hypergeometric k-function defined by

$$ {}_pF_q^k\left[\begin{matrix}(\alpha_1,\mathrm{k})\cdots(\alpha_p,\mathrm{k}) \\ \\ (\beta_1,\mathrm{k})\cdots(\beta_q,\mathrm{k})\end{matrix} \quad ; \quad z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\cdots(\alpha_p)_{n,k}}{(\beta_1)_{n,k}\cdots(\beta_q)_{n,k}}\frac{z^n}{n!} $$

The series representation of the generalized Lommel Wright k-function defined as

$$ J_{\aleph,\hbar,k}^{\psi,m}(\mathrm{z}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma_k(\hbar+k+nk)^m\Gamma_k(\aleph+\hbar+n\psi+k)}\left(\frac{z}{2}\right)^{2n+\frac{\aleph+2\hbar}{k}} \tag{1.1} $$

Where $z \in C/(-\infty,0], m \in N, \aleph, \hbar \in C, \psi > 0$ and $k \in R^+$.

Here $\Gamma_k(z)$ is the k-Gamma function introduced by Diaz and Pariguan [12] given as

$$ \Gamma_k(z) = \lim_{n\to\infty}\frac{(n)!\,k^n\,(nk)^{\frac{z}{k}-1}}{(z)_{n,k}} $$

with k-Pochhammer symbol

$$ (\mathrm{z})_{n,k} = \mathrm{z}(\mathrm{z}+k)(\mathrm{z}+2k)\ \ldots.(\mathrm{z}+(n-1)k)\ z \in C,\ \ k \in R^+ $$

The classical Euler Gamma function and Gamma k-function are related with following relation

$$ \Gamma_k(\mathrm{z}) = k^{\frac{z}{k}-1}\Gamma\left(\frac{\mathrm{z}}{k}\right) $$

for k=1 generalized Lommel Wright k-function reduce in generalized Lommel Wright function. For m=1 reducing in generalized Bessel Maitland k-function. For m=k=1 reducing in Bessel Maitland function. For m=ψ=k=1 and $\hbar = 0$ reducing in claasical Bessel function.

II. WHITTAKER FUNCTION

In 1930, Whittaker showed that it is possible to express some special functions are in terms of a new function suggested by him. i.e. the Whittaker function. Two Whittaker functions are applied today and they are defined using the kummar confluent hypergeometric function previously is given as[4,11]:

$$ W_{(k,m)}(x) = \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-m-k\right)}M_{(k,m)}(x) + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m-k\right)}M_{(k,-m)}(x) $$

$$ M_{\kappa,m}(x) = \exp\left(-\frac{x}{2}\right)\,x^{m+\frac{1}{2}}\ {}_1F_1\left(m-k+\frac{1}{2},1+2m,x\right) $$

The following known results of Mathai and Saxena (Mathai and Saxena,1973) as follows:

$$ \int_0^{+\infty} x^{\delta-1}\exp(x/2)\,W_{(\eta,\alpha)}(x)\,\mathrm{d}x = \frac{\Gamma(1/2\pm\alpha+\delta)\Gamma(-\eta-\delta)}{\Gamma(1/2\pm\alpha-\eta)} \tag{2.1} $$

$$ \int_0^{+\infty} x^{\delta-1}\exp(-x/2)\,M_{(\eta,m)}(x)\,\mathrm{d}x = \frac{\Gamma(2m+1)\Gamma(m+\delta+1/2)\Gamma(\eta-\delta)}{\Gamma(m-\delta+1/2)\Gamma(m+\eta+1/2)} \tag{2.2} $$

$$ \int_0^{+\infty} x^{\delta-1}W_{(\eta,\alpha)}(x)W_{(-\eta,\alpha)}(x)\,\mathrm{d}x = \frac{\Gamma\left(\frac{\delta+1}{2}\pm\alpha\right)\,\Gamma(\delta+1)}{2\,\Gamma\left(1+\frac{\delta}{2}\pm\eta\right)} \tag{2.3} $$

III. MAIN RESULT

Theorem 3.1 Let $z \in C/(-\infty,0], m \in N, \aleph, \hbar \in C, \psi > 0$ *and* $k \in R^+$ then the product of the Whittaker function and the generalized Lommel Wright k-function is defined as

$$ \int_0^{+\infty} z^{\rho-1}\exp(az/2)W_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)\mathrm{d}z = \frac{k^{2-\left(\frac{1}{2}\pm\alpha-\eta\right)}\left(\frac{\omega}{2(a)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}}{(a)^\rho\Gamma\left(\frac{1}{2}\pm\alpha-\eta\right)} $$

$$\times\ {}_2\Psi_{m+1}^{k}\left[\begin{matrix}\left(\frac{1}{2}k\pm\alpha k+\rho k+\theta(\aleph+2\hbar),2\theta\mathrm{k}\right)(-\eta k-\rho k-\theta(\aleph+2\hbar),-2\theta\mathrm{k}) & ; & \left(-\frac{\omega^2}{4(a^2)^\theta}\right)\\ (\hbar+\mathrm{k},\mathrm{k})\cdots(\hbar+\mathrm{k},\mathrm{k})\ (\aleph+\hbar+k,\psi) & & \end{matrix}\right] \quad (3.1)$$

Proof: Letting a z=x, a dz=dx as z→ 0, x→ 0 and z→∞, x→∞ and using (1.1) in the integrand of (3.1) which is followed uniform convergence of the involved series under the given conditions, we get

$$\int_0^{+\infty} z^{\rho-1}\exp(az/2)W_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)\mathrm{dz}=\frac{(\omega/2)^{\frac{\aleph+2\hbar}{k}}}{a^{\rho+\theta\left(\frac{\aleph+2\hbar}{k}\right)}}\times\sum_{n=0}^{\infty}\frac{\left(-\frac{\omega^2}{4(a^2)^\theta}\right)^n}{\Gamma_k(\hbar+k+nk)^m\Gamma_k(\aleph+\hbar+n\psi+k)}$$

$$\times\int_0^{+\infty} x^{\rho+\theta\left(2n+\frac{\aleph+2\hbar}{k}\right)-1}\exp(x/2)\,W_{(\eta,\alpha)}(x)dx$$

So that

$$\int_0^{+\infty} z^{\rho-1}\exp(az/2)W_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)\mathrm{dz}\ =\frac{k^{2-\left(\frac{1}{2}\pm\alpha-\eta\right)}(\omega/2)^{\frac{\aleph+2\hbar}{k}}}{\Gamma\left(\frac{1}{2}\pm\alpha-\eta\right)a^{\rho+\theta\left(\frac{\aleph+2\hbar}{k}\right)}}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma_k\left(\frac{1}{2}k\pm\alpha k+\rho k+\theta(2nk+\aleph+2\hbar)\right)\Gamma_k(-\eta k-\rho k-\theta(2nk+\aleph+2\hbar))}{\Gamma_k(\hbar+k+nk)^m\Gamma_k(\aleph+\hbar+n\psi+k)}\left(-\frac{\omega^2}{4(a^2)^\theta}\right)^n$$

Now using (2.1) in the above equation, we get

$$\int_0^{+\infty} z^{\rho-1}\exp(az/2)W_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)\mathrm{dz}=\frac{k^{2-\left(\frac{1}{2}\pm\alpha-\eta\right)}\left(\frac{\omega}{2(a)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}}{(a)^\rho\Gamma\left(\frac{1}{2}\pm\alpha-\eta\right)}$$

$$\times\ {}_2\Psi_{m+1}^{k}\left[\begin{matrix}\left(\frac{1}{2}k\pm\alpha k+\rho k+\theta(\aleph+2\hbar),2\theta\mathrm{k}\right)(-\eta k-\rho k-\theta(\aleph+2\hbar),-2\theta\mathrm{k}) & ; & \left(-\frac{\omega^2}{4(a^2)^\theta}\right)\\ (\hbar+\mathrm{k},\mathrm{k})\cdots(\hbar+\mathrm{k},\mathrm{k})(\aleph+\hbar+k,\psi) & & \end{matrix}\right]$$

Which gives our statement (3.1). this completes the proof of theorem.

Theorem 3.2. Let $z\in C/(-\infty,0], m\in N, \aleph,\hbar\in C, \psi>0\ and\ k\in R^+$ then the product of the Whittaker function and the generalized Lommel Wright k-function is defined in (1.1) is given as

$$\int_0^{+\infty} z^{\rho-1}\exp(-\,az/2)M_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)\mathrm{dz}\ =\frac{k^{1-\eta}\left(\frac{\omega}{2(ak)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}\Gamma(2\alpha+1)}{(ak)^\rho\Gamma\left(\frac{1}{2}+\alpha+\eta\right)}$$

$$\times\ {}_2\Psi_{m+2}^{k}\left[\begin{matrix}\left(\alpha k+\rho k+\theta(\aleph+2\hbar)+\frac{1}{2}k,2\theta\mathrm{k}\right)(\eta k-\rho k-\theta(\aleph+2\hbar),-2\theta\mathrm{k}) & ; & \left(-\frac{\omega^2}{4(a^2k^2)^\theta}\right)\\ (\hbar+\mathrm{k},\mathrm{k})\cdots(\hbar+\mathrm{k},\mathrm{k})(\aleph+\hbar+k,\psi)\left(\alpha k-\rho k-\theta(\aleph+2\hbar)+\frac{1}{2}k,-2\theta\mathrm{k}\right) & & \end{matrix}\right] \quad (3.3.2)$$

Proof: Letting a z=x, a dz=dx as z→ 0, x→ 0 and z→∞, x→∞ and using (1.1) in the integrand of (3.2) which is followed uniform convergence of the involved series under the given conditions, we get

$$\int_0^{+\infty} z^{\rho-1}\exp(-az/2)M_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)\mathrm{dz}$$

$$=\frac{(\omega/2)^{\frac{\aleph+2\hbar}{k}}}{a^{\rho+\theta\left(\frac{\aleph+2\hbar}{k}\right)}}\times\sum_{n=0}^{\infty}\frac{\left(-\frac{\omega^2}{4(a^2)^\theta}\right)^n}{\Gamma_k(\hbar+k+nk)^m\Gamma_k(\aleph+\hbar+n\psi+k)}\times\int_0^{+\infty} x^{\rho+\theta\left(2n+\frac{\aleph+2\hbar}{k}\right)-1}\exp(-x/2)\,M_{(\eta,\alpha)}(x)dx$$

$$\int_0^{+\infty} z^{\rho-1}\exp(-az/2)M_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)\mathrm{dz}\ =\frac{k^{1-\eta}\left(\frac{\omega}{2(ak)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}\Gamma(2\alpha+1)}{(ak)^\rho\Gamma\left(\frac{1}{2}+\alpha+\eta\right)}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma_k\left(\alpha k+\rho k+\theta(2nk+\aleph+2\hbar)+\frac{1}{2}k\right)\Gamma_k(\eta k-\rho k-\theta(2nk+\aleph+2\hbar))}{\Gamma_k(\hbar+k+nk)^m\Gamma_k(\aleph+\hbar+n\psi+k)\Gamma_k\left(\alpha k-\rho k-\theta(2nk+\aleph+2\hbar)+\frac{1}{2}k\right)}\left(-\frac{\omega^2}{4(a^2k^2)^\theta}\right)^n$$

Now using (2.2) in the above equation, we get

$$\int_0^{+\infty} z^{\rho-1}\exp(-az/2)M_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)\mathrm{dz}=\frac{k^{1-\eta}\left(\frac{\omega}{2(ak)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}\Gamma(2\alpha+1)}{(ak)^\rho\Gamma\left(\frac{1}{2}+\alpha+\eta\right)}$$

$$\times\ {}_2\Psi_{m+2}^{k}\left[\begin{matrix}\left(\alpha k+\rho k+\theta(\aleph+2\hbar)+\frac{1}{2}k,2\theta k\right)(\eta k-\rho k-\theta(\aleph+2\hbar),-2\theta k) & ; & \left(-\frac{\omega^2}{4(a^2k^2)^\theta}\right)\\ (\hbar+k,k)\cdots(\hbar+k,k)(\aleph+\hbar+k,\psi)\left(\alpha k-\rho k-\theta(\aleph+2\hbar)+\frac{1}{2}k,-2\theta k\right) & & \end{matrix}\right]$$

Which gives our statement (3.2). this completes the proof of theorem.

Theorem 3.3. Let $z\in C/(-\infty,0], m\in N, \aleph,\hbar\in C, \psi>0\ and\ k\in R^+$ then the product of the Whittaker function and the generalized Lommel Wright k-function is defined in (1.1) is given as

$$\int_0^{+\infty} z^{\rho-1}W_{(\eta,\alpha)}(az)W_{(-\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)dz=\frac{k^{\frac{1}{2}\pm\alpha\pm\eta}\left(\frac{\omega}{2(ak)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}}{2(ak)^\rho}$$

$$\times\ {}_2\Psi_{m+2}^{k}\left[\begin{matrix}\left(\frac{\rho k+\theta(\aleph+2\hbar)+k}{2}\pm\alpha k,\theta k\right)(\rho k+\theta(\aleph+2\hbar)+k,2\theta k) & ; & \left(-\frac{\omega^2}{4(a^2k^2)^\theta}\right)\\ (\hbar+k,k)\cdots(\hbar+k,k)(\aleph+\hbar+k,\psi)\left(k+\frac{\rho k+\theta(\aleph+2\hbar)}{2}\pm\eta k,\theta k\right) & & \end{matrix}\right]\qquad(3.3)$$

Proof: Letting a z=x, a d z=dx as z→ 0, x→ 0 and z→∞, x→∞ and using (1.1) in the integrand of (3.3) which is followed uniform convergence of the involved series under the given conditions, we get

$$\int_0^{+\infty} z^{\rho-1}W_{(\eta,\alpha)}(az)W_{(-\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)dz$$

$$=\frac{(\omega/2)^{\frac{\aleph+2\hbar}{k}}}{a^{\rho+\theta\left(\frac{\aleph+2\hbar}{k}\right)}}\times\sum_{n=0}^{\infty}\frac{\left(-\frac{\omega^2}{4(a^2)^\theta}\right)^n}{\Gamma_k(\hbar+k+nk)^m\Gamma_k(\aleph+\hbar+n\psi+k)}\int_0^{+\infty}x^{\rho+\theta\left(2n+\frac{\aleph+2\hbar}{k}\right)-1}W_{(\eta,\alpha)}(x)W_{(-\eta,\alpha)}(x)dx$$

So that

$$\int_0^{+\infty} z^{\rho-1}W_{(\eta,\alpha)}(az)W_{(-\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)dz=$$

$$\frac{k^{\frac{1}{2}\pm\alpha\pm\eta}\left(\frac{\omega}{2(ak)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}}{(ak)^\rho}\times\sum_{n=0}^{\infty}\frac{\Gamma_k\left(\frac{\rho k+\theta(2nk+\aleph+2\hbar)+k}{2}\pm\alpha k\right)\Gamma_k(\rho k+\theta(2nk+\aleph+2\hbar)+k)}{\Gamma_k(\hbar+k+nk)^m\Gamma_k(\aleph+\hbar+n\psi+k)\,2\Gamma_k\left(k+\frac{\rho k+\theta(2nk+\aleph+2\hbar)}{2}\pm\eta k\right)}\left(-\frac{\omega^2}{4(a^2k^2)^\theta}\right)^n$$

Now using (2.3) in the above equation, we get

$$\int_0^{+\infty} z^{\rho-1}W_{(\eta,\alpha)}(az)W_{(-\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi,m}(\omega z^\theta)dz=\frac{k^{\frac{1}{2}\pm\alpha\pm\eta}\left(\frac{\omega}{2(ak)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}}{2(ak)^\rho}$$

$$\times\ {}_2\Psi_{m+2}^{k}\left[\begin{matrix}\left(\frac{\rho k+\theta(\aleph+2\hbar)+k}{2}\pm\alpha k,\theta k\right)(\rho k+\theta(\aleph+2\hbar)+k,2\theta k) & ; & \left(-\frac{\omega^2}{4(a^2k^2)^\theta}\right)\\ (\hbar+k,k)\cdots(\hbar+k,k)(\aleph+\hbar+k,\psi)\left(k+\frac{\rho k+\theta(\aleph+2\hbar)}{2}\pm\eta k,\theta k\right) & & \end{matrix}\right]$$

Which gives our statement (3.3). this completes the proof of theorem.

3.4 SPECIAL CASES : In this section, we obtain some integral formulas involving Lommel Wright k-function as follows.

Corollary 3.4.1. Let $z\in C/(-\infty,0], \aleph,\hbar\in C, \psi>0\ and\ k\in R^+$ and m=1 in (3.1) then we obtain:

$$\int_0^{+\infty} z^{\rho-1}\exp(az/2)W_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi}(\omega z^\theta)dz=\frac{k^{2-\left(\frac{1}{2}\pm\alpha-\eta\right)}\left(\frac{\omega}{2(a)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}}{(a)^\rho\Gamma\left(\frac{1}{2}\pm\alpha-\eta\right)}$$

$$\times\ {}_2\Psi_2^{k}\left[\begin{matrix}\left(\frac{1}{2}k\pm\alpha k+\rho k+\theta(\aleph+2\hbar),2\theta k\right)(-\eta k-\rho k-\theta(\aleph+2\hbar),-2\theta k) & ; & \left(-\frac{\omega^2}{4(a^2)^\theta}\right)\\ (\hbar+k,k)\ \ (\aleph+\hbar+k,\psi) & & \end{matrix}\right]$$

Corollary 3.4.2. Let $z\in C/(-\infty,0], \aleph,\hbar\in C, \psi>0\ and\ k\in R^+$ and m=1 in (3.2) then we obtain:

$$\int_0^{+\infty} z^{\rho-1}\exp(-az/2)M_{(\eta,\alpha)}(az)\,J_{\aleph,\hbar,k}^{\psi}(\omega z^\theta)dz=\frac{k^{1-\eta}\left(\frac{\omega}{2(ak)^\theta}\right)^{\frac{\aleph+2\hbar}{k}}\Gamma(2\alpha+1)}{(ak)^\rho\Gamma\left(\frac{1}{2}+\alpha+\eta\right)}$$

$$\times\ {}_2\Psi_3^{k}\left[\begin{matrix}\left(\alpha k+\rho k+\theta(\aleph+2\hbar)+\frac{1}{2}k,2\theta k\right)(\eta k-\rho k-\theta(\aleph+2\hbar),-2\theta k) & ; & \left(-\frac{\omega^2}{4(a^2k^2)^\theta}\right)\\ (\hbar+k,k)\ \ (\aleph+\hbar+k,\psi)\left(\alpha k-\rho k-\theta(\aleph+2\hbar)+\frac{1}{2}k,-2\theta k\right) & & \end{matrix}\right]$$

Corollary 3.4.3. Let $z \in C/(-\infty,0], \aleph, \hbar \in C, \psi > 0$ and $k \in R^+$ and m=1 in (3.3) then we obtain:

$$\int_0^{+\infty} z^{\rho-1} W_{(\eta,\alpha)}(az) W_{(-\eta,\alpha)}(az) J^{\psi}_{\aleph,\hbar,k}(\omega z^{\theta}) dz = \frac{k^{\frac{1}{2}\pm\alpha\pm\eta}\left(\frac{\omega}{2(ak)^{\theta}}\right)^{\frac{\aleph+2\hbar}{k}}}{2(ak)^{\rho}}$$

$$\times\ {}_2\Psi_3^k\left[\begin{matrix}\left(\frac{\rho k+\theta(\aleph+2\hbar)+k}{2}\pm\alpha k, \theta k\right)(\rho k+\theta(\aleph+2\hbar)+k, 2\theta k) \\ (\hbar+k,k)\ (\aleph+\hbar+k,\psi)\left(k+\frac{\rho k+\theta(\aleph+2\hbar)}{2}\pm\eta k, \theta k\right)\end{matrix};\ \left(-\frac{\omega^2}{4(a^2k^2)^{\theta}}\right)\right]$$

Corollary 3.4.4. Let $z \in C/(-\infty,0], m \in N, \aleph, \hbar \in C$ and $\psi > 0$ and k=1 in (3.1) then we obtain:

$$\int_0^{+\infty} z^{\rho-1}\exp(az/2) W_{(\eta,\alpha)}(az) J^{\psi,m}_{\aleph,\hbar}(\omega z^{\theta}) dz$$

$$= \frac{\left(\frac{\omega}{2(a)^{\theta}}\right)^{\aleph+2\hbar}}{(a)^{\rho}\Gamma\left(\frac{1}{2}\pm\alpha-\eta\right)} \times\ {}_2\Psi_{m+1}\left[\begin{matrix}\left(\frac{1}{2}\pm\alpha+\rho+\theta(\aleph+2\hbar), 2\theta\right)(-\eta-\rho-\theta(\aleph+2\hbar), -2\theta) \\ (\hbar+1,1)\cdots(\hbar+1,1)\ (\aleph+\hbar+1, \psi)\end{matrix};\ \left(-\frac{\omega^2}{4(a^2)^{\theta}}\right)\right]$$

Corollary 3.4.5. Let $z \in C/(-\infty,0], m \in N, \aleph, \hbar \in C$ and $\psi > 0$ and k=1 in (3.2) then we obtain:

$$\int_0^{+\infty} z^{\rho-1}\exp(-az/2) M_{(\eta,\alpha)}(az) J^{\psi,m}_{\aleph,\hbar}(\omega z^{\theta}) dz = \frac{\left(\frac{\omega}{2(a)^{\theta}}\right)^{\aleph+2\hbar}\Gamma(2\alpha+1)}{(a)^{\rho}\Gamma\left(\frac{1}{2}+\alpha+\eta\right)}$$

$$\times\ {}_2\Psi_{m+2}\left[\begin{matrix}\left(\alpha+\rho+\theta(\aleph+2\hbar)+\frac{1}{2}, 2\theta\right)(\eta-\rho-\theta(\aleph+2\hbar), -2\theta) \\ (\hbar+1,1)\cdots(\hbar+1,1)\ (\aleph+\hbar+1,\psi)\left(\alpha-\rho-\theta(\aleph+2\hbar)+\frac{1}{2}, -2\theta\right)\end{matrix};\ \left(-\frac{\omega^2}{4(a^2)^{\theta}}\right)\right]$$

Corollary 3.4.6. Let $z \in C/(-\infty,0], m \in N, \aleph, \hbar \in C$ and $\psi > 0$ and k=1 in (3.3) then we obtain:

$$\int_0^{+\infty} z^{\rho-1} W_{(\eta,\alpha)}(az) W_{(-\eta,\alpha)}(az) J^{\psi,m}_{\aleph,\hbar}(\omega z^{\theta}) dz$$

$$= \frac{\left(\frac{\omega}{2(a)^{\theta}}\right)^{\aleph+2\hbar}}{2(a)^{\rho}} \times\ {}_2\Psi_{m+2}\left[\begin{matrix}\left(\frac{\rho+\theta(\aleph+2\hbar)+1}{2}\pm\alpha, \theta\right)\ (\rho+\theta(\aleph+2\hbar)+1, 2\theta) \\ (\hbar+1,1)\cdots(\hbar+1,1)(\aleph+\hbar+1,\psi)\left(1+\frac{\rho+\theta(\aleph+2\hbar)}{2}\pm\eta, \theta\right)\end{matrix};\ \left(-\frac{\omega^2}{4(a^2)^{\theta}}\right)\right]$$

Corollary 3.4.7. Let $z \in C/(-\infty,0], \aleph \in C$ and m = k = ψ = 1 and $\hbar = 0$ in (3.1) then we obtain:

$$\int_0^{+\infty} z^{\rho-1}\exp(az/2) W_{(\eta,\alpha)}(az) J_{\aleph}(\omega z^{\theta}) dz = \frac{\left(\frac{\omega}{2(a)^{\theta}}\right)^{\aleph}}{(a)^{\rho}\Gamma\left(\frac{1}{2}\pm\alpha-\eta\right)} \times\ {}_2\Psi_2\left[\begin{matrix}\left(\frac{1}{2}\pm\alpha+\rho+\theta\aleph, 2\theta\right)(-\eta-\rho-\theta\aleph, -2\theta) \\ (1,1)\ \ (\aleph+1,1)\end{matrix};\ \left(-\frac{\omega^2}{4(a^2)^{\theta}}\right)\right]$$

Corollary 3.4.8. Let $z \in C/(-\infty,0], \aleph \in C$ and m = k = ψ = 1 and $\hbar = 0$ in (3.2) then we obtain:

$$\int_0^{+\infty} z^{\rho-1}\exp(-az/2) M_{(\eta,\alpha)}(az) J_{\aleph}(\omega z^{\theta}) dz = \frac{\left(\frac{\omega}{2(a)^{\theta}}\right)^{\aleph}\Gamma(2\alpha+1)}{(a)^{\rho}\Gamma\left(\frac{1}{2}+\alpha+\eta\right)} \times\ {}_2\Psi_3\left[\begin{matrix}\left(\alpha+\rho+\theta\aleph+\frac{1}{2}, 2\theta\right)(\eta-\rho-\theta\aleph, -2\theta) \\ (1,1)\ (\aleph+1,1)\left(\alpha-\rho-\theta\aleph+\frac{1}{2}, -2\theta\right)\end{matrix};\ \left(-\frac{\omega^2}{4(a^2)^{\theta}}\right)\right]$$

Corollary 3.4.9. Let $z \in C/(-\infty,0], \aleph \in C$ and m = k = ψ = 1 and $\hbar = 0$ in (3.3) then we obtain:

$$\int_0^{+\infty} z^{\rho-1} W_{(\eta,\alpha)}(az) W_{(-\eta,\alpha)}(az) J_{\aleph}(\omega z^{\theta}) dz = \frac{\left(\frac{\omega}{2(a)^{\theta}}\right)^{\aleph}}{2(a)^{\rho}} \times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\rho+\theta\aleph+1}{2}\pm\alpha, \theta\right)(\rho+\theta\aleph+1, 2\theta) \\ (1,1)\ (\aleph+1,1)\left(1+\frac{\rho+\theta\aleph}{2}\pm\eta, \theta\right)\end{matrix};\ \left(-\frac{\omega^2}{4(a^2)^{\theta}}\right)\right]$$

IV. Conclusion:

The integration of product of Whittaker function with respect to the generalized Lommel Wright k-function were derived. When we take k = 1 we also deduce the results for Lommel wright function.

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