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An Analysis of q-Baskakov–Szasz Operators

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Abstract : *This research study provides an expansion of the classic Baskakov-Sz´asz operators, integrating an unique limitation classified as q, which provides a modified method to the initial operators. This modification broadens the traditional theory to encompass q-calculus, an extension of standard calculus that incorporates a new parameter q. This extension allows for a broader exploration of operators within a more comprehensive mathematical framework. Our approach builds upon the fundamental principles important in approximation theory, but adapts them to align with q-calculus, offering new perspectives and techniques for managing functions and sequences in this context.* We provide a detailed analysis of the moment computations associated with these q-operators, which is crucial for comprehending their behavior and setting the foundation for different approximation methods.

Moments play a key role in approximation theory by offering insights into how well the operators can represent certain classes of functions accurately. By conducting this computation, we create a structure for delving further into the properties of the operators, such as their rates of convergence and limits on errors. In addition to calculating moments, we also analyze the modulus of continuity, which smoothness in functions. This continuity measure is important in approximation theory as it helps determine how accurately an operator can approximate a function, especially considering its smoothness. Our study shows that the q-Baskakov-Sz´asz operators possess favorable properties for approximation, making them useful tools for function approximation in the field of q-calculus. As specific variables approach infinity, we create mathematical q-operators. This mathematical representation serves as a useful resource for the extended-term behaviors of these operators and provides insight into their effectiveness in predicting outcomes over prolonged durations.

Our work has revealed several outstanding issues that require additional exploration. These issues involve exploring different q-operators, investigating their potential applications in various fields, and conduct-ing a more comprehensive analysis of their convergence properties. At-tending to these unconcluded issues are going to lead the way for im-provident in the research study of q-calculus and its real-world effects in estimation concept.

key words: Asymptotic expressions, Baskakov-Sz´asz operators, Continuity modulus, Quantum integers, q-based analogues,

I. INTRODUCTION

Gupta [6] laid out the theoretical Baskakov-Sz´asz type operators, building upon the existing knowledge of these operators. These opera-tors offer a broad set of characteristics that expand the scope of their applications in approximation theory, effectively extending the capabilities of the standard Baskakov and Sz´asz operators. The introduction of these operators by Gupta opened up new avenues for exploring con-vergence properties across different functional spaces and delving into function approximation studies.

$$P_m(u,z) = m\sum_{h=0} b_{m,h}(z)\int_0^\infty c_{m,h}(y)u(y)dy, z \in [0,\infty)$$

Whereas

$$b_{m,h}(z) = \binom{m+h-1}{h}\frac{z^h}{(1+z)^{m+h}}$$

$$c_{m,h}(y) = e^{-my}\frac{(my)^h}{h!}$$

It is evident from reference [6] that the operators mentioned only duplicate functions that do not vary. Extensive studies on q operators have been conducted in the last decade. Aral et.al [1], Abel and Gupta [2], Abel [3], Aslan [4], de la cal [5], Kumar [7] Mishra [8], Usta [9] and others have contributed to recent research in thas heal. Io begin, we will examine various q-calculus symbols documented in [4] and [2]. This section indicates that q represents a real number such that 0 is leses than q which is less than 1 .

For $m \in \mathbb{N}$,

$$[m]_q := \frac{1-q^m}{1-q}, [m]_q! := \begin{cases}[m]_q[m-1]_q \cdots [1]_q, m = 1,2, \ldots \\ 1, m = 0\end{cases}$$

The q-Beta integral is defined by kumar [7]

$$\Gamma_q(y) = \int_0^{\frac{1}{1-\alpha}} z^{y-1} E_\psi(-qz) d_\psi z, y > 0$$

Solution to the functional equation given:

$$\Gamma_q(y+1) = |y|_\psi \Gamma_q(y), \Gamma_q(1) = 1$$

The operators known as the q-Baskakov operators by Abel [2] are defined for a function f in the continuous interval from 0 to infinity, where q is a positive number and m is any positive integer.

$$G_{m,f}(u,z) = \sum_{h=0}^{\infty} \begin{bmatrix} m+h-1 \\ h \end{bmatrix}_q q^{\frac{bb-1}{2}} \frac{z^h}{(1+z)_q^{n+b}} u\left(\frac{[h]_q}{q^{h-1}[m]_q}\right)$$
$$= \sum_{h=0}^{\infty} b_{m,h}^q(z) u\left(\frac{[h]_q}{q^{h-1}[n]_q}\right)$$

where

$$(1+z)_q^m := \begin{cases}(1+z)(1+qz)\ldots(1+q^{m-1}z), m = 1,2, \ldots \\ 1, m = 0\end{cases}$$

The binomial coefficients for the variable q are determined by

$$\begin{bmatrix} m \\ h \end{bmatrix}_q = \frac{[m]_q!}{| h]_q! [m-h]_q{}^2}, 0 \leq h \leq m$$

Remark 1. The first few instances of the q-Baskakov operators are, offering an initial glimpese into their makeup and use. These examples lay the groundwork for comprehending the wider capabilities of the q Baskakov operators, which play a significant role in different contexts within approximation theory.

$$G_{m_q}(1,z) = 1, G_{m_q,q}(y,z) = z, G_{m,q}(y^2,z) = z^2 + \frac{z}{[m]_q}\left(1 + \frac{1}{q}z\right).$$

The operators $P_{\mathrm{m}}(u,z)$ include different sorts of busis features with both summation and combination procodures, illustrating the amazing capacity of recreating straight features. This particular imspired us to discover these drivers in better extent. Structure on this, we currently present the q-analog of these kinds of operators, using a brand-new standpoint on their actions within the system of q-calculus.

$$P_m^q(u,z) = [m]_q \sum_{h=0}^{\infty} b_{m,h}^q(z) \int_0^{q/(1-q^-)} q^{-h-1} c_{m,h}^q(y) u(yq^{-h}) dqy$$

belongs to the interval [0, infinity) and (refer to [6] for more information) are provided by

$$b_{m,h}^q(z) = \begin{bmatrix} m+h-1 \\ h \end{bmatrix}_q q^{\frac{m-1}{2}} \frac{z^h}{(1+z)_q^{m+h}}, c_{m,h}^q(y) = E_q(-[m]_q t) \frac{([m]_q l)}{[h]_q!}$$

When $q = 1$, the formerly discussed operators lower to their timeless equivalents, expecially the operators reviewed in formula (1). The purpose of this paper is to perform a thorough evuluation of these operators, especifically checking out a regional estimate thesis that explains their habits in particular areas. Furthermore, we research the price of merging of these freshly presented q-operators, in addition to their calculated estimate buildings, which supply undenstanding right into their efficiency when put on features with particular development limitations or habits at infinity. Via this evaluation, we intend to clarify exactly how these operators preserve or beset their estimation abilities in numerous settings.

2. Moment Point Computation q-analogue of Baskakov-Szász operators

In statistical analysis, moment estimation plays a crucial role in characterizing the form and dispersion of data sets. When applied to mathematical operators, like the q-analogue of Baskakov-Száwa operators, moment estimation assesses their effectiveness in approximating functions. By precisely estimating uncover key attributes of these operators, including their convergence rates, precision, and limits of error.

Lemma 1.Gupta [6] These mathematical identities can be confirmed within the established framework as foundational building blocks for the creation and examination of related mathematical constructs. They embody crucial connections that are vital for understanding the behavior and attributes of these constructs, including their ability to converge, approximate, and exhibit other important traits that are pertinent to the specific problem being addressed.

Within the framework of q-calculus and related mathematical disciplines, verifying the validity of these equalities is crucial for systematically investigating novel operator typess, thereby guarnanteeing the preservation of favornble attributes such us stability, smooth, or contimity. Consequently, these equalities play a dual role, serving both as immediate results and as fundamental building blocks that inform and direct mubsequent research, enabling the comstruction of more intricate mathematical frame works.

Lemma1:
(i) $Pz(1,z) = 1$
(ii) $P_m^P(y,z) = z + \frac{b}{|m|_n}$
(iii) $P_m^q(y^2,z) = z^2\left(1 + \frac{1}{q|m|_k}\right) + \frac{z}{|m|_q}(1 + q(q+2)) + \frac{q^2(1+q)}{|m|_q|}$.

Proof. Applying the operator $P_{总}$ to the constant function 1 , the linear function y, and the quadratic function y^2, each function undergoes a unique trunuformation. These changes are sugnificant no they showcuse the operator's behavior when acting upon basic, essential functions. The outcomes of these alterations provide valuable information about the characteristios of the operator and its ability to approximate effectively.

$$P_m^q(1,z) = [m]_q \sum_{h=0}^{\infty} b_{m,h}^q(z) \int^{q/(1-q)} q^{-b-1} \frac{\left([m]_q t\right)^h}{[h]_q!} E_q\left(-[m]_q t\right) d_q y$$

Substituting $[m]_q y = qx$ and using (2), we have

$$\begin{aligned} P_m^\gamma(1,z) &= [m]_q \sum_{h=0}^{\infty} b_{m,h}^q(z) \int_0^{1/(1-q)} q^{-h-1} \frac{(qx)^h}{[h]_q^2} E_q(-qx) \frac{qd}{[m]_q} \\ &= \sum_{h=0}^{\infty} b_{m,h}^q(z) \frac{\Gamma_q(h+1)}{[h]_q} \\ &= G_{m_f}(1,z) = 1 \end{aligned}$$

In this situation, $G_{\text{mq}}(u,z)$ represents the q-Baskakov operator, as introduced in equation (3) before. This operator serves as a q-version of the traditional Baskakov operator commonly applied in approximation theory. It relies on various factons like m, q, m, and z, with m and q usually influencing the type of approximation, while u and z are variables affecting the operator's function. The expression "Following this, we will proceed" suggests that additional outcomes or statements will come next, probably concerning the q-Baskakov operator to showcase its characteristics, compute moments, or establish a particular theorem regarding the operator's performance. Theme subsequent actions will expand on the definition outlined in equation (3).

$$P_m^q(y,z) = [m]_q \sum_{h=0}^{\infty} b_{m,h}^q(z) \int_0^{q/(1-q^{-c})} q^{-b-1} \frac{\left([m]_q y\right)^h}{[h]_q!} E_q\left(-[m]_q \ell\right) y q^{-b} d_q y$$

By replacing $[m]_q y = qx$ once more and applying (2), along with the information from Remark 1, we obtain the result

$$[h+1]_q = []_q + q^h.$$

$$\begin{aligned} P_m^q(y,z) &=| m]_q \sum_{h=0}^{\infty} P_{m,h}^q(x) \int_0^{1/(1-q)} q^{-h-1} \frac{(qx)^{k+1}}{[h]_q!\,[m]_q} E_q(-qx) \frac{qdx}{[m]_q q} \\ &= \sum_{h=0}^{\infty} b_{m,h}^q(z) \frac{[h+1]_q}{[m]_q q^{h-1}} \\ &= \sum_{h=0}^{\infty} b_{m,h}^q(z) \frac{[h]_q + q^k}{[m]_q q^{h-1}} \\ &= V_{m,q}(y,z) + \frac{q}{[m]_q} G_{m,h}(1,z) = z + \frac{q}{[m]_q} \end{aligned}$$

The statement "Lastly, we calculate the second moment in the fol lowing manner" the required calculations have been carried out, the subsequent statement will provide an approximation for this second moment. This approximation plays a vital role in comprehending the precision and convergence characteristics of the operator, especially in the context of function approximation. It aids in establishing the error margins and the efficiency of the operator.

$$P_m^Q(y^2,z) = [m]_q \sum_{h=0}^{\infty} b_{m,b}^q(z) \int_0^{q/(1-q^-)} q^{-h-1} \frac{\left([m]_q y\right)^h}{[h]_q^q} E_q\left(-[m]_q t\right) y^2 q^{-2h} d_q y$$

Again substituting $[m]_q y = qx$, uning (2), $[h+1]_q = [h]_q + q^h, [h+2]_q = [h]_q + q^h + q^{h+1}$ and Remark 1, we have

$$\begin{aligned} P_m^q(y^2,z) &= [m]_q \sum_{h=0}^{\infty} b_{m,h}^q(z) \int_0^{1/h1-q)} q^{-h-1} \frac{(qx)^{h+2}}{[h]_q^t \mid m]_q^2} E_q(-qx) q^{-2h} \frac{q d_q x}{[m]_q} \\ &= \sum_{h=0}^{\infty} b_{m,h}^q(z) \frac{[h+1]_q [h+2]_q}{[m]_q q^{qh-2}} = \sum_{h=0}^{\infty} b_{m,h}^q(z) \frac{\left(|h|_q + q^h\right)\left([h_q + q^h + q^{h+1}\right)}{[m]_q q^{2h-2}} \\ &= G_{m_q q}(y^2,z) + \frac{2q+q^2}{[m]_q} G_{m_g}(y,z) + \frac{q^2(1+q)}{[m]_q^2} \\ &= z^2\left(1 + \frac{1}{q[m]_q}\right) + \frac{z}{[m]_q}(1 + q(q+2)) + \frac{q^2(1+q)}{[m]_q^2}. \end{aligned}$$

Lemma 2. Let $q \in (0,1)$, then for $z \in [0,\infty)$, we have

$$P_m^Q(y-z,z) = \frac{q}{[m]_q}, P_m^q((y-z)^2,z) = \frac{q^3(1+q) + [m]_q z(z+q+q^2)}{q[m]_q^2}$$

Remark 2. At $q = 1$, the sequence $P_m(u,z)$ simplifies to a more traditional or well-known form of the operators, as formulated by Aral et al. The moments of this sequence, represented as $P_m(u,z)$ when viewed as operators, can be interpreted in the following way:

$$P_m(y-z,z) = \frac{1}{m}, P_m((y-z)^2,z) = \frac{2+mx(2+z)}{m^2}$$

Remark 3. For every $q \in (0,1)$ we have

$$P_m^q((y-z)^2,z) \leq \frac{2}{q[m]_q}\left(\varphi^2(z) + \frac{1}{[m]_q}\right)$$

where $\varphi^2(z) = z(1+z), z \in [0,\infty)$.

3. Fundamental Mathematical Theorem

Let $Dp[0,\infty)$ real -valued, continuous, and bounded functions u 8 defined on the interval $[0,\infty)$. This space consists of functions that are continuous for all $y \in [0,\infty)$ and whose values remain bounded over this interval. The norm $\|-|$ on the space $D_p(0,\infty)$ is defined by the supreme (or maximum aboolute value) of the function over the interval $[0,\infty)$, which is given by: $\|u\| = \sup_{y\in(p,\infty)} |u(y)|$ This norm meanures the largest absolute value that the function u attains over the entire interval. Thus, the space $D_p[0,\infty)$, equipped with this norm, forms a normed vector apace of bounded continuous functions.

$$\|u\| = \sup_{0\leq st<\infty} |u(t)|$$

The Peetre's K-functional is defined as follows:

$$K_2(u,\Omega) = \inf[\{\|u-g\| + \Omega\|g''\|: g \in W_\infty^2\}$$

where $W_\infty^2 = \{g \in Dp[0,\infty): g', g'' \in C_p[0,\infty)\}$. By Abel and Leviatan [3], there exists a positive constant $D > 0$ such that $K_2(u,\Omega) \leq D\omega_2(u,\Omega^{1/2}), \Omega > 0$

where the second order modulus of smoothness is given by

$$\omega_2(u,\sqrt{\Omega}) = \sup_{0<k\leq\sqrt{00s;<\infty}} \sup|u(z+2k) - 2u(z+k) + u(z)|$$

Also, for $u \in D_p[0,\infty)$ the usual modulus of continuity is given by

$$\omega(u,\Omega) = \sup_{0<k\leq b} \sup_{0\leq<\infty} |u(z+k) - u(z)|$$

Theorem 1. Let $u \in D_P[0,\infty)$ and $0 < q < 1$. Then for all $z \in [0,\infty)$ and $m \in N$, there exists an absolute constant $D > 0$ such thant

$$\left|P_m^q(u,z) - u(z)\right| \leq D\omega_2\left(u,\frac{\Omega_m(z)}{\sqrt{q[m]_q}}\right) + \omega\left(u,\frac{q}{|\,m]_q}\right)$$

where $\Omega_m^2(z) = \varphi^2(z) + \frac{1}{|mJ_q}$.

Proof. Let us introduce the auxiliary operators P_m^Z defined by

$$\bar{P}q(u,z) = P(u,z) - u\left(z + \frac{q}{|m|_q}\right) + u(z)$$

$z \in [0,\infty)$. The operators $\overline{\mathcal{P}}_m^q$ are linear and preserve the linear functions:

$$P_m^q(y-z,z) = 0$$

Let $g \in W^2$. From Taykor's series

$$g(y) = g(z) + g'(z)(y-z) + \int_z^y (y-v)g''(v)dv, y \in [0,\infty)$$

and (6), we get

$$P_m^y(g,z) = g(z) + P_m\left(\int_z^y (y-v)g^n(v)dv, z\right)$$

Hence, by (5) one has

$$\left|P_m^g(g,z) - g(z)\right| \leq \left|P_m^g\left(\int_z^y (y-v)g^u(v)dv, z\right)\right| + \left|\int_z^{z+\frac{1}{\sim}}\left(z + \frac{q}{ml_p} - v\right)g^u(v)dv\right|$$

$$\leq P_m^g\left(\left|\int^u \right| y - v||g^n(v)|dv|, z\right)$$

$$+ \int^{z+\frac{1}{m\cdot n}} \left|z + \frac{q}{|\,m]_q} - v\right|\left|z + \frac{q}{|\,m]_q} - v\right| |g^n(v)|dv$$

Using Remark 3, we obtain

$$P\underline{m}((y-z)^2,z) + \left(\frac{q}{|\,m]_q}\right)^2 \leq \frac{2}{q\,|\,m]_q}\left(\varphi^2(z) + \frac{1}{[m]_q}\right) + \left(\frac{q}{|\,m]_q}\right)^2$$

Then, by (7), we get

$$\left|P_m^q(g,z) - g(z)\right| \leq \frac{2}{q\,|\,m]_q}\Omega_m^2(z)|g^n|$$

On the other hand, by (3.1), (3.,4) and Lemma 1, we have

$$\left|P_m^p(u,z)\right| \leq |P_m m^p(u,z)| + 2\|u| \leq \|u\|P_m^y(1,z) + 2\|u\| \leq 3|u\|.$$

Now (3.1), (3.4), and (3.5) simplify

$$|P\underline{q}(u,z)-u(z)| \leq \left|\overline{D_m^q}(u-g,z)-(u-g)(z)\right| + \left|\overline{P_m^q}(g,z)-g(z)\right| + \left|u\left(z+\frac{q}{[m]_q}\right)-u(z)\right| \leq 4\left\|u-g\right| + \frac{2}{q[m]_q}\Omega_m^2(z)\|g^n\| + \left|u\left(z+\frac{q}{\mid m]_q}\right)-u(z)\right|$$

Hence taking minimum on the right hand side over all $g \in W^2$, we get

$$\left|P_m^q(u_1z)-u(x)\right| \leq 4K_2\left(u,\frac{1}{q[m]_q}\Omega_m(z)\right)+\omega\left(u,\frac{q}{[m]_q}\right)$$

In view of the property of K functional, for every $q \in (0,1)$ we get

$$\left|P_m^q(u,z)-u(z)\right| \leq C_{\omega_2}\left(u,\frac{\Omega_n(z)}{\sqrt{q\mid m]_q}}\right)+\omega\left(u,\frac{q}{[m]_q}\right)$$

This completes the proof of the theorem.

Theorem 2. Let $u \in D_{x^2}[0,\infty), q \in (0,1)$ and $\omega_{a+1}(f,\Omega)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a>0$. Then, we have

$$\left\|P_m^q(u)-u\right\|_{D\{0,u\}} \leq \frac{D_1}{q[m]_q}+2\omega_{u+1}\left(u\sqrt{\frac{D_2}{q[m]_q}}\right)$$

where D_1 and D_2 are certain constants.

Proof. For $z \in [0,a]$ and $y>a+1$, since $y-z>1$, we have

$$|u(y)-u(z)| \leq M_l(2+z^2+y^2)$$
$$\leq M_z(2+3z^2+2(y-z)^2) \leq 6M_z(1+a^2)(y-z)^2$$

For $z \in [0,a]$ and $y \leq a+1$, we have

$$|u(y)-u(z)| \leq \omega_{a+1}(u,|y-z|) \leq \left(1+\frac{|y-z|}{\Omega}\right)\omega_{a+1}(u,\Omega)$$

with $\Omega>0$. From (3.6) and (3.7) we can write

$$|u(y)-u(z)| \leq 6M_z(1+a^2)(y-z)^2+\left(1+\frac{|y-z|}{\Omega}\right)w_{w+1}(u,\Omega)(3.8)$$

for $z \in [0,a]$ and $y \geq 0$. Thus

$$\left|P_m^q(u,z)-u(z)\right| \leq P_m^q(|u(y)-u(z)|_1z) \leq 6M_u(1+a^2)P_m^P((y-z)^2,z) + w_{u+1}(u,\Omega)\left(1+\frac{1}{\Omega}\mathcal{D}_m^g((y-z)^2,z)\right)^{\frac{1}{4}}$$

Hence, by Schwartz's inequality and Remark 3 , for every $q \in (0,1)$ and $z \in [0,a]$

$$|P_m^\Omega(u,z)-u(z)| \leq \frac{12M_u(1+a^2)}{q[m]_q}\left(\varphi^2(z)+\frac{1}{[m]_q}\right) + \omega_{a+1}(u,\Omega)\left(1+\frac{1}{\Omega}\sqrt{\frac{2}{q[m]_q}\left(\varphi^2(z)+\frac{1}{[m]_q}\right)}\right) \leq \frac{D_1}{q[m]_q} + \omega_{a+1}(u,\Omega)\left(1+\frac{1}{\Omega}\sqrt{\frac{D_2}{q[m]_q}}\right)$$

By taking $\Omega=\sqrt{\frac{D_2}{q[m]_q}}$ we get the assertion of our theorem.

4. Complex Moments and Asymptotic Expansion

Lemma 3 [5]. Let $0 < q < 1$, we have

$$G_{m,q}(y^3,z) = \frac{1}{[m]_q} z + \frac{1+2q}{q^2}\frac{[m+1]_q}{[m]_q^2} z^2 + \frac{1}{q^3}\frac{[m+1]_q[n+2]_q}{[m]_q^2} z^3$$

$$G_{m,q}(y^4,z) = \frac{1}{[m]_q^3} x + \frac{1}{q^3}(1+3q+3q^2)\frac{[m+1]_q}{[m]_q^3} x^2 + \frac{1}{q^5[2]_q}(1+3q+5q^2+3q^3)\frac{[m+1]_q[m+2]_q}{[m]_q^3} x^3 + \frac{1}{q^6[2]_q[3]_q[4]_q}(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)\frac{[m+1]_q[m+2]_q[m+3]_q}{[m]_q^3} z^4$$

Now we present higher order moments for our operators (4). Lemma 4 [6]. Let $0 < q < 1$, we have

$$P_m^q(y^3,z) = \frac{1}{q^3}\frac{[m+1]_q[m+2]_q}{[m]_q^2} zx^3 + \left(\frac{1+2q}{q^2}\frac{[m+1]_q}{[m]_q^2} + \frac{q(3+2q+q^2)}{[m]_q} + \frac{3+2q+q^2}{[m]_q^2}\right) z^2 + \left(\frac{q^5+3q^4+5q^3+5q^2+3q+1}{[m]_q^2}\right) z + \frac{q^3(1+2q+2q^2+q^3)}{[m]_q^3}$$

$$P_m^q(y^4,z) = G_m^q(y^4,z) + \frac{q(4+3q+2q^2+q^3)}{[m]_q} G_m^q(y^3,z) + \frac{q^2\left([3]_q[4]_q + (2+q)(2+2q+2q^2+q^3) + [2]_q\right)}{[m]_q^2} G_m^q(y^2,z) + \frac{q^3\left((2+q)[3]_q[4]_q + [2]_q(2+2q+2q^2+q^3)\right)}{[m]_q^3} G_m^q(y,z) + \frac{q^4[2]_q[3]_q[4]_q}{[m]_q^4}$$

We can use Lemma 3, to obtain the exact value.

Proof. Py simple computation, we have

$$\begin{aligned}
P_m^q(y^3,z) &= \sum_{h=0}^{\infty} b_{m,h}^q(z)\frac{[h+1]_q[h+2]_q[h+3]_q}{[m]_q^3 q^{3h-3}} \\
&= \sum_{h=0}^{\infty} b_{m,h}^q(z)\frac{\left([h]_q+q^h\right)\left([h]_q+q^h+q^{h+1}\right)\left([h]_q+q^h+q^{h+1}+q^{h+2}\right)}{[m]_q^3 q^{3h-3}} \\
&= G_m^q(y^3,z) + \frac{q(3+2q+q^2)}{[m]_q} G_m^q(y^2,z) + \frac{q^2(3+4q+3q^2+q^3)}{[m]_q^2} G_m^q(y,z) + \frac{q^3(1+2q+2q^2+q^3)}{[m]_q^3} \\
&= \frac{1}{[m]_q^2} z + \frac{1+2q}{q^2}\frac{[m+1]_q}{[m]_q^2} z^2 + \frac{1}{q^3}\frac{[m+1]_q[m+2]_q}{[m]_q^2} z^3 + \frac{q(3+2q+q^2)}{[m]_q}\left(z^2 + \frac{z}{[m]_q} + \frac{z^2}{q[m]_q}\right) \\
&\quad + \frac{q^2(3+4q+3q^2+q^3)}{[m]_q^2} z + \frac{q^3(1+2q+2q^2+q^3)}{[m]_q^3} \\
&= \frac{1}{q^3}\frac{[m+1]_q[m+2]_q}{[m]_q^2} z^3 + \left(\frac{1+2q}{q^2}\frac{[m+1]_q}{[m]_q^2} + \frac{q(3+2q+q^2)}{[m]_q} + \frac{3+2q+q^2}{[m]_q^2}\right) z^2 \\
&\quad + \left(\frac{q^5+3q^4+5q^3+5q^2+3q+1}{[m]_q^2}\right) + \frac{q^3(1+2q+2q^2+q^3)}{[m]_q^3}
\end{aligned}$$

Similarly

$$P_m^q(y^4,z) = \sum_{h=0}^{\infty} b_{m,h}^q(z)\frac{[h+1]_q[h+2]_q[h+3]_q[h+4]_q}{[q]_q^4 q^{4k-4}}$$
$$= \sum_{h=0}^{\infty} b_{m,h}^q(z)\frac{([h]_q+q^h)([h]_q+q^h+q^{h+1})([h]_q+q^h+q^{k+1}+q^{h+2})([h]_q+q^h+q^h}{[m]_q^4 q^{4h-4}}$$
$$= \sum_{h=0}^{\infty} b_{m,h}^q(z)\frac{[h]_q^4+[h]_q^3 q^h(4+3q+2q^2+q^3)+[h]_q^2 q^{2h}([3)q[4)_q+(2+q)(2+2q+}{[m]_q^4 q^{4h-4}}$$
$$+\sum_{h=0}^{\infty} b_{m,h}^q(z)\frac{[h]_q q^{3h}\left((2+q)[3]q[4]_q+[2]_q(2+2q+2q^2+q^3)\right)+q^{4h}[2]_q[3]q[\text{l }4]_q}{[m]_q^4 q^{4k-4}}G_m^q$$
$$+\frac{q(4+3q+2q^2+q^3)}{[m]_q}G_m^q(y^3,z)+\frac{q^2([3]_q[4]_q+(2+q)(2+2q+2q^2+q^3)+[2]_q}{[m]_q^2}$$
$$+\frac{q^3\left((2+q)[3]_q[4]_q+[2]_q(2+2q+2q^2+q^3)\right)}{[m]_q^3}G_m^q(y,z)+\frac{q^4[2]_q[3]_q[4]q}{[m]_q^4}.$$

Remark 4. As the operators defined by (4) are linear, we have

$$P_m^q((y-z)^4,z) = P_m^q(y^4,z) - 4zP_m^q(y^3,z) + 6z^2P_m^q(y^2,z) - 4z^3P_m^q(y,z) + z^4$$

We consider the following classes of functions:

$$D_n[0,\infty) := \left\{u \in D[0,\infty) : \exists M_u > 0 |u(z)| < M_f(1+z^n) \text{ and } \|u\|_n := \sup_{z\in[0,\infty)} \frac{|u(z)|}{1+z^u}\right\}$$
$$D_n^*[0,\infty) := \left\{u \in D_n[0,\infty) : \lim_{z\to\infty} \frac{|u(z)|}{1+z^n} < \infty\right\}, n \in \mathbb{N}$$

Theorem 3. Let $q_m \in (0,1)$. Then the sequence $\{P_m^{q_m}(u)\}$ converges to u uniformly on $[0,B]$ for each $u \in D_2^*[0,\infty)$ if and only if $\lim_{m\to\infty} q_m = 1$.

Theorem 4. Assume that $q_m \in (0,1), q_m \to 1$ and $q_m^m \to$ a as $m \to \infty$. For any $u \in D_2^*[0,\infty)$ such that $u', u'' \in D_2^*[0,\infty)$ the following equality holds

$$\lim_{m\to\infty} [m]_{q_m}\left(P_m^{q_m}(u,z) - u(z)\right) = u'(z) + (z^2+2z)u''(z)$$

uniformly on any $[0,B], B > 0$.

Proof. Let $u, u', u'' \in D_2^*[0,\infty)$ and $z \in [0,\infty)$ be fixed. Py the Taylor formula we may write

$$u(y) = u(z) + u'(z)(u-z) + \frac{1}{2}u''(z)(y-z)^2 + r(y,z)(y-z)^2 \tag{4.1}$$

where $r(y,z)$ is the Peano form of the remainder, $r(y,z) \in D_2^*[0,\infty)$ and $\lim_{y\to z} r(y,z) = 0$. Applying $P_m^{q_m}$ to (4.1) we obtain

$$[m]_{q_m}\left(P_m^{q_m}(u,z) - u(z)\right) = [m]_{q_m}P_m^{q_m}(y-z,z)f'(z) + [m]_{q_m}P_m^{q_m}((y-z)^2,z)\frac{u''(z)}{2} + [m]_{q_m}P_m^{q_m}(r(y,z)(y-z)^2,z)$$

By the Cauchy-Schwartz inequality, we have

$$P_m^{q_m}(r(y,z)(y-z)^2,z) \le \sqrt{P_m^{q_m}(r^2(y,z),z)}\sqrt{P_m^{q_m}((y-z)^4,z)} \tag{4.2}$$

Observe that $r^2(y,z) = 0$ and $r^2(.,z) \in D_2^*[0,\infty)$. Then it follows from Theorem 3 and Lemma 4, that

$$\lim_{m\to\infty} P_m^{q_m}(r^2(y,z),z) = r^2(y,z) = 0 \tag{4.3}$$

uniformly with respect to $z \in [0,B]$. Now from (4.2),(4.3) and Lemma 2, we get immediately

$$\lim_{m\to\infty} [m]_{q_m}P_m^{q_m}(r(y,z)(y-z)^2,z) = 0$$

Then we get the following

$$\lim_{z\to\infty}[z]_{q_m}\left(P_m^{q_m}(u,z)-u(z)\right)$$
$$=\lim_{m\to\infty}[m]_{q_m}\left(u'(z)P_m^{q_n}((y-z),x)+\frac{1}{2}u''(z)P_m^{q_m}((y-z)^2,z)+P_m^{q_m}(r(y,z)(y-z)^2;z)\right)$$
$$=u'(z)+\left(\frac{z^2+2z}{2}\right)u''(z)$$

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