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EXISTENCE RESULT FOR THIRD ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT: In this Research paper, by using hybrid fixed point theorem we shall discuss the existence result for Third order nonlinear functional differential equations in $\mathcal{R}_+$.
KEYWORDS: Functional differential equation, Banach Algebras, Hybrid fixed point theorem ,Banach Space and Existence result.

1. INTRODUCTION:

The nonlinear differential equations have been studied extensively in the Literature by several authors for various aspects of the solutions. The study of nonlinear fractional differential equations had been made extensively in the literature by several authors all over the world and now it has become the core part of the nonlinear analysis. [26-31].

The theory of Differential and Integral equations is rapidly developing using the tools of Topology, Functional Analysis and Fixed point theory. This is particularly true for problems in the related fields of Engineering, and Mathematical Physics. There are number of applications of differential and integral equations of integer and fractional orders in Electrochemistry, Viscoelasticity, Control theory, Electromagnetism and Porous media etc. [7-15, 34-37]

In this paper we will study the existence the solution of third order nonlinear functional differential equation.

We consider the following third order nonlinear functional differential equation:

$$\left.\begin{aligned} \mathcal{D}^3\left[\frac{x(t)}{f\big(t,x(\theta(t))\big)}\right] &= g[t,x(\mu(t))], \qquad t\in\mathcal{R}_+ \\ x(0) &= 0 \end{aligned}\right\} \tag{1.1}$$

Where, $f(t,x):\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}-\{0\}$, $g(t,x):\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}$ and $\theta,\mu:\mathcal{R}_+\to\mathcal{R}$

Here the solution of nonlinear differential equations (1.1) we mean a function $x\in BC(\mathcal{R}_+,\mathcal{R})$ such that:

(i) The function $t\to\left[\frac{x(t)}{f(t,x(\theta(t)))}\right]$ is bounded and continuous for each $x\in\mathcal{R}$.

(ii) x satisfies (1.1)

2. PRELIMINARIES:

Let $X=BC(\mathcal{R}_+,\mathcal{R})$ be the space of bounded real valued continuous function on $\mathcal{R}_+$ and S be a subset of X. Let a mapping $\mathcal{A}:X\to X$ be an operator and consider the following operator equation in X, namely,

$$x(t)=(\mathcal{A}x)(t),\ \text{for all } t\in\mathcal{R}_+ \tag{2.1}$$

Definition 2.1[34]: Let (X, d) be the metric space and $a\in X$ and for some real number $r>0$ the set $B_r[a]=\{x\in X: d(x,a)\le r\}$ is called closed ball centered at a with radius r.

Definition 2.2[23]: Let X be a Banach space. A mapping $\mathcal{A}:X\to X$ is called Lipschitz if there is a constant $\alpha>0$ such that, $\|\mathcal{A}x-\mathcal{A}y\|\le\alpha\|x-y\|$ for all $x,y\in X$. If $\alpha<1$, then $\mathcal{A}$ is called a contraction on X with the contraction constant α.

Definition 2.3[19]:.An operator $\mathcal{U}$ on a Banach space X into itself is called compact if for any bounded subset S of X, $\mathcal{U}(S)$ is relatively compact subset of X. If $\mathcal{U}$ is continuous and compact, then it is called completely continuous on X.

Definition 2.4[19]: Let X be a Banach space with the norm $\|\cdot\|$ and let $\mathcal{U}:X\to X$ be an operator (in general nonlinear). Then $\mathcal{U}$ is called

i. Compact if $\mathcal{U}(X)$ is relatively compact subset of X.
ii. Totally bounded if $\mathcal{U}(S)$ is totally bounded subset of X for any bounded subset S of X.
iii. Completely continuous if it is continuous and totally bounded operator on X

It is clear that every compact operator is totally bounded but the converse need not be true.

Definition 2.5[22]: Let $f \in \mathcal{L}^1[0,\mathcal{T}]$ and $\alpha > 0$. The Riemann – Liouville fractional derivative of order ζ of real function f is defined as

$$\mathcal{D}^{\zeta} f(t) = \frac{1}{\Gamma(1-\zeta)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\zeta}} ds \quad , \quad 0 < \zeta < 1$$

Such that $\mathcal{D}^{-\zeta} f(t) = I^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{f(s)}{(t-s)^{1-\zeta}} ds$ respectively.

Definition 2.6[22]: The Riemann-Liouville fractional integral of order $\zeta \in (0,1)$ of the function $f \in \mathcal{L}^1[0,\mathcal{T}]$ is defined by the formula:

$$I^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{f(s)}{(t-s)^{1-\zeta}} ds \, , \quad t \in [0, \mathrm{T}]$$

Where $\Gamma(\zeta)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ζ defined by

$$\mathcal{D}^{\zeta} = \frac{d^{\zeta}}{dt^{\zeta}} = \frac{d}{dt} \circ I^{1-\zeta}$$

Definition 2.7[34]: $\Gamma(n+1) = n\,!$

Theorem 2.1 [6] :(Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $\mathcal{C}(\mathcal{R}_+, \mathcal{R})$, then it has a convergent subsequence.

Theorem 2.2[6]: A metric space X is compact iff every sequence in X has a convergent subsequence.

Theorem 2.3[5, 6, and 18]: Let S be a non-empty, bounded and closed-convex subset of the Banach space X and let $\mathcal{A}: X \to X$ and $\mathcal{B}: S \to X$ are two operators satisfying

a) $\mathcal{A}$ is Lipschitz with a lipschitz constant α,
b) $\mathcal{B}$ is completely continuous, and
c) $\mathcal{A}x\mathcal{B}x \in S$ for all $x \in S$, and
d) $\alpha M < 1$,Where $M = \|\mathcal{B}(S)\|: \sup\{\|\mathcal{B}x\|: x \in S\}$.

Then the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution in S.

3. EXISTENCE THEORY:

Now we want the solution of (1.1) in the space $BC(\mathcal{R}_+, \mathcal{R})$of bounded and continuous real valued functions defined on $\mathcal{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $BC(\mathcal{R}_+, \mathcal{R})$by,$\|x\| = sup\{|x(t)|: t \in \mathcal{R}_+\}$, $(xy)(t) = x(t)y(t)$, $t \in \mathcal{R}_+$ (3.1)

Clearly, $BC(\mathcal{R}_+, \mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ we denote the space of Lebesgue-integrable function in $\mathcal{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by $\|x\|_{\mathcal{L}} = \int_0^{\infty} |x(t)| dt$ (3.2)

Definition 3.1[6]: A mapping $g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is Caratheodory if:

i) $t \to g(t,x)$ is measurable for each $x \in \mathcal{R}$ and
ii) $x \to g(t,x)$ is continuous almost everywhere for $t \in \mathcal{R}_+$.

Furthermore a Caratheodory function g is $\mathcal{L}^1$ −Caratheodory if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that$|g(t,x)| \le h_r(t) a.e.$ $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| \le r$

Finally a Caratheodory function g is $\mathcal{L}_X^1$ − Caratheodory if:

iv) There exists a function $h \in \forall \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| \le h(t)$, $a.e.$ $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$

For convenience, the function h is referred to as a bound function for g.

4. MAIN RESULT:

We need following hypothesis for existence of solution of third order nonlinear functional differential equation (TNFDE) (1.1)

(B_1) The functions $\theta, \mu: \mathcal{R}_+ \to \mathcal{R}$ are continuous.

(B_2) The function $f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and bounded with bound F= $sup_{(t,x(\theta(t))) \epsilon \mathcal{R}_+ \times \mathcal{R}} \left| f\left(t, x(\theta(t))\right) \right|$there exist a bounded function $l: \mathcal{R}_+ \to \mathcal{R}$ with bound L satisfying

$\left| f\left(t, x(\theta(t))\right) - f\left(t, y(\theta(t))\right) \right| \le l(t)\{|x(\theta(t)) - y(\theta(t))|\}$ for $t \in \mathcal{R}_+$

for all $x, y \in \mathcal{R}$.

(B_3) The function $g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is satisfying caratheodory condition with continuous function $h(t): \mathcal{R}_+ \to \mathcal{R}$ such that $g(t,x) \le h(t)$ $\forall t \in \mathcal{R}_+$ and $x, y \in \mathcal{R}$.

(B_4) The function $v: \mathcal{R}_+ \to \mathcal{R}$ defined by the formulas $v(t) = \int_0^t (t-s)^2 h(s)\, ds$ is bounded on $\mathcal{R}_+$ and vanish at infinity, that is $\lim_{t \to \infty} v(t) = 0$.

Remark.4.1: Note that the **(B_3)** and **(B_4)** hold, then there exists a constant $K_1 > 0$ such that $K_1 = \sup\{v(t): t \epsilon \mathcal{R}_+\}$

Lemma 4.1: Suppose that $\zeta \in (0,1)$ and the function f, g satisfying TNFDE (1.1) then x is the solution of the TNFDE (1.1) if and only if it is the solution of integral equation

$$x(t) = \left[f\left(t, x(\theta(t))\right)\right] \frac{1}{(2)} \left[\int_0^t (t-s)^2 \; g\left(s, x(\mu(s))\right) ds\right], t \in \mathcal{R}_+ \qquad (4.1)$$

Proof: Integrating equation (1.1) we get,

$$I\mathcal{D}^3 \left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right]_0^t = I\left[g\left(s, x(\mu(s))\right)\right]$$

$$\mathcal{D}^2 \left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right]_0^t = I\left[g\left(s, x(\mu(s))\right)\right]$$

$$\mathcal{D}^2 \left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right] = I\left[g\left(s, x(\mu(s))\right)\right]$$

Again integrating, we get

$$\mathcal{D} \left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right]_0^t = I^2\left[g\left(s, x(\mu(s))\right)\right]$$

$$\mathcal{D} \left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right] = I^2\left[g\left(s, x(\mu(s))\right)\right]$$

Again integrating, we get

$$\left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right] = I^3\left[g\left(s, x(\mu(s))\right)\right]$$

$$x(t) = \left[f\left(t, x(\theta(t))\right)\right]\left[I^3\left[g\left(s, x(\mu(s))\right)\right]\right]$$

$$x(t) = \left[f\left(t, x(\theta(t))\right)\right] \frac{1}{(3-1)!} \int_0^t \quad (t-s)^2 \quad g\left(s, x(\mu(s))\right) ds$$

$$x(t) = \left[f\left(t, x(\theta(t))\right)\right] \frac{1}{(2)} \left[\int_0^t (t-s)^2 \; g\left(s, x(\mu(s))\right) ds\right], t \in \mathcal{R}_+$$

Since $\int_0^t f(t) dt^n = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds$, Where $n = 0,1,2,3, \dots \dots \dots$

Conversely differentiate (4.1) of order 3 w.r.to t, we get,

$$\mathcal{D}^3 \left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right] = \mathcal{D}^3 \; \frac{1}{(2)} \left[\int_0^t (t-s)^2 \; g\left(s, x(\mu(s))\right) ds\right]$$

$$\mathcal{D}^3 \left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right] = \mathcal{D}^3 \left[\frac{1}{\Gamma(3)} \int_0^t (t-s)^{3-1} g\left(s, x(\mu(s))\right) ds\right]$$

$$\mathcal{D}^3 \left[\frac{x(t)}{f\left(t, x(\theta(t))\right)}\right] = \quad g\left(s, x(\mu(t))\right)$$

Theorem 4.2: Assume that condition **(B_1) - (B_4)** hold. Further if $LK_1 < 1$, where K_1 is defined in remark (4.1). Then TNFDE (1.1) has a solution in the space $BC(\mathcal{R}_+, \mathcal{R})$.

Proof: By a solution of TNFDE (1.1) we mean a continuous function $x: \mathcal{R}_+ \to \mathcal{R}$ that satisfies TNFDE (1.1) on $\mathcal{R}_+$. Let $X = BC(\mathcal{R}_+, \mathcal{R})$ and define a subset $B_r[0]$ of X as $B_r[0] = \{x \in X: \|x\| \le r\}$.where r satisfies the inequality, $\frac{1}{2} F K_1 \le r$.

Let $X = BC(\mathcal{R}_+, \mathcal{R})$ be Banach Algebra of all bounded continuous real-valued function on $\mathcal{R}_+$ with the norm $\|x\| = sup|x(t)|, t \in \mathcal{R}_+$ (4.2)
Now the TNFDE (1.1) is equivalent to the

$$x(t) = \left[f\left(t, x(\theta(t))\right)\right] \frac{1}{(2)} \left[\int_0^t (t-s)^2 \ g\left(s, x(\mu(s))\right) ds\right], t \in \mathcal{R}_+$$

Let us define the two mappings $\mathcal{A}: X \to X$
and $\mathcal{B}: B_r[0] \to X$ by

$$\mathcal{A}x(t) = f\left(t, x(\theta(t))\right), t \in \mathcal{R}_+ \quad (4.3)$$

$$\mathcal{B}x(t) = \frac{1}{2} \left[\int_0^t (t-s)^2 \ g\left(s, x(\mu(s))\right) ds\right], t \in \mathcal{R}_+ \quad (4.4)$$

Thus from the TNDE (1.1), we obtain the operator equation as follows:

$$x(t) = \mathcal{A}x(t)\mathcal{B}x(t), \ t \in \mathcal{R}_+ \quad (4.5)$$

If the operator $\mathcal{A}$ and $\mathcal{B}$ satisfy all the hypothesis of theorem (2.3), then the operator equation (4.5) has a solution on $B_r[0]$.

Step I: Firstly we show that $\mathcal{A}$ is Lipschitz on X . Let $x, y \in X$; then

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq \left|f\left(t, x(\theta(t))\right) - f\left(t, y(\theta(t))\right)\right|$$

$$\leq l(t)\{|x(\theta(t)) - y(\theta(t))|\}$$

$\leq L|x(t) - y(t)|$ for all $t \in \mathcal{R}_+$

Taking supremum over t we get,

$\|\mathcal{A}x - \mathcal{A}y\| \leq L\|x - y\|$ for all $x, y \in B_r[0]$

Thus, $\mathcal{A}$ is Lipchitz on X with Lipschitz constant L.
Step II: Secondly we show that $\mathcal{B}$ is completely continuous operator on $B_r[0]$using standard argument such as those in Granas at [18], it can be shown that $\mathcal{B}$ is continuous operator on $B_r[0]$.
To do this, let us fix arbitrary $\epsilon > 0$ and take $x, y \in B_r[0]$ such that $\|x - y\| \leq \epsilon$.

$$|\mathcal{B}x(t) - \mathcal{B}y(t)| = \left|\begin{matrix}\frac{1}{2} \left[\int_0^t (t-s)^2 \ g\left(s, x(\mu(s))\right) ds\right] - \\ \frac{1}{2} \left[\int_0^t (t-s)^2 \ g\left(s, y(\mu(s))\right) ds\right]\end{matrix}\right|$$

$$\leq \frac{1}{2}\left|\int_0^t (t-s)^2 g\left(s, x(\mu(s))\right) ds\right| + \frac{1}{2}\left|\int_0^t (t-s)^2 g\left(s, y(\mu(s))\right) ds\right|$$

$$\leq \frac{1}{2}\int_0^t (t-s)^2 h(s)\, ds + \frac{1}{2}\int_0^t (t-s)^2 \, h(s)\, ds$$

$$\leq 2\ \frac{1}{2}\int_0^t \ (t-s)^2 \, h(s)\, ds \ ,$$

$$\leq \int_0^t \ (t-s)^2 \, h(s) ds \leq v(t) \quad \text{(by Hypothesis } B_4 \text{)}$$

$|\mathcal{B}x(t) - \mathcal{B}y(t)| \leq \epsilon.$ As $v(t) \leq \epsilon$

Thus $\mathcal{B}$ is continuous.

Step III: Now we will show that $\mathcal{B}$ is compact on $\mathcal{B}(B_r[0])$

First we prove that every sequence $\{ \mathcal{B}x_n\}$ in $\mathcal{B}(B_r[0])$has uniformly bounded sequence and $\{ \mathcal{B}x_n\}$ is equicontinuous set in $B_r[0]$.
Since $g\left(t, x(\mu(t))\right)$ is $\mathcal{L}_X^1 -$ caratheodary, we have

$$|\mathcal{B}x_n(t)| = \frac{1}{2}\left|\int_0^t (t-s)^2 \, g\left(s, x_n(\mu(s))\right) ds\right|$$

$$\leq \frac{1}{2}\int_0^t (t-s)^2 \left|g\left(s, x_n(\mu(s))\right)\right| ds$$

$\leq \frac{1}{2}\int_0^t (t-s)^2\, h(s)ds$

$\leq v(t)$ (By Hypothesis B_4)

Taking supremum over t, we obtain, $\|\mathcal{B}x_n\| \leq K_1$ for all $x \in B_r[0]$

Where, $K_1 = sup_{t\in\mathcal{R}_+}\{v(t)\}$

This shows that $\{\mathcal{B}x_n\}$ is uniformly bounded sequence in $\mathcal{B}(B_r[0])$

To show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence, let $t_1, t_2 \in [0,T]$ be arbitrary. Then for any $x \in B_r[0]$

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = \left| \frac{1}{2}\int_0^{t_2}(t_2-s)^2 g\left(s, x_n(\mu(s))\right)ds - \frac{1}{2}\int_0^{t_1}(t_1-s)^2 g\left(s, x_n(\mu(s))\right)ds \right|$$

$$= \frac{1}{2}\left|\int_0^{t_2}(t_2-s)^2 h(s)ds - \int_0^{t_1}(t_1-s)^2 h(s)ds\right|$$

$$\leq \frac{1}{2}|v(t_2) - v(t_1)|$$

The right hand side of the above inequality doesn't depend on x and tends to zero as $t_1 \to t_2$.Therefore$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0$ as $t_1 \to t_2$.

If $t_1, t_2 \geq T$ then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = \left| \frac{1}{2}\int_0^{t_2}(t_2-s)^2 g\left(s, x_n(\mu(s))\right)ds - \frac{1}{2}\int_0^{t_1}(t_1-s)^2 g\left(s, x_n(\mu(s))\right)ds \right|$$

$$= \frac{1}{2}\left|\int_0^{t_2}(t_2-s)^2 h(s)ds - \int_0^{t_1}(t_1-s)^2 h(s)ds\right|$$

$$\leq \frac{1}{2}\left|\int_0^{t_2}(t_2-s)^2 h(s)ds\right| + \left|\int_0^{t_1}(t_1-s)^2 h(s)ds\right| \quad \text{(by Hypothesis } B_4 \text{)}$$

$$\leq \tfrac{1}{2}v(t_2) + \tfrac{1}{2}v(t_1) \leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \leq \epsilon \text{ as } t_1 \to t_2$$

If $t_1, t_2 \in \mathcal{R}_+$ then we have

$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)|$

If $t_1 \to t_2$, then $t_1 \to T$ *and* $T \to t_2$

Therefore $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| \to 0$ $|\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)| \to 0$

So $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0$ as $t_1 \to t_2$

Hence, $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in $\mathcal{B}(B_r[0])$ so $\mathcal{B}(B_r[0])$ is relatively compact by Arzela-Ascoli theorem. By definition 2.4 $\mathcal{B}$ is compact which gives, $\mathcal{B}$ is compact and continuous operator on $B_r[0]$.

Thus $\mathcal{B}$ is completely continuous on $B_r[0]$

Step IV: To show $x = \mathcal{A}x\mathcal{B}y \in B_r[0]$

Let $x, y \in B_r[0]$ such that $x = \mathcal{A}x\mathcal{B}x$

$|x(t)| = |\mathcal{A}x(t)\mathcal{B}x(t)|$

$\leq |\mathcal{A}x(t)||\mathcal{B}x(t)|$

$\leq \left|f\left(t,x\big(\theta(t)\big)\right)\right|\left|\frac{1}{2}\int_0^t (t-s)^2 g\left(s,x\big(\mu(s)\big)\right)ds\right|$

$\leq \frac{1}{2}\left|f\left(t,x\big(\theta(t)\big)\right)\right|\int_0^t (t-s)^2\left|g\left(s,x\big(\mu(s)\big)\right)\right|ds$

$\leq F\frac{1}{2}\int_0^t (t-s)^2 h(s)ds \leq \frac{1}{2}Fv(t)$(by Hypothesis H_8)

Taking supremum over ton $\mathcal{R}_+$, we obtain $\|\mathcal{A}x\mathcal{B}x\| \leq \frac{1}{2}FK_1$, $\forall x \in B_r[0]$

That is we have, $\|x\| = \|\mathcal{A}x\mathcal{B}x\| \leq r$, $\forall x \in B_r[0]$.

which gives $x = \mathcal{A}x\mathcal{B}y \in B_r[0]$

Hence assumption(c) of theorem (2.3) is proved.

Step V: Also we have

$M = \|\mathcal{B}(B_r[0])\| = sup\{\|\mathcal{B}x\|: x \in (B_r[0])\}$

$= sup\left\{sup_{t\in\mathcal{R}_+}\left[\frac{1}{2}\int_0^t (t-s)^2 g\left(s,x\big(\mu(s)\big)\right)ds\right] : x \in B_r[0]\right\}$

$\leq sup\left\{sup_{t\in\mathcal{R}_+}\left[\frac{1}{2}\int_0^t (t-s)^2 h(s)ds\right] : x \in B_r[0]\right\}$

$\leq \frac{1}{2}sup\{sup_{t\in\mathcal{R}_+}[v(t)]: x \in B_r[0]\}$

$\leq \frac{1}{2}K_1$

and therefore $LM = \frac{1}{2}LK_1 < 1$

Thus the condition (d) of theorem (2.3) is satisfied.

Hence all the conditions of theorem (2.3) are satisfied and therefore the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution in $B_r[0]$. As a result, the TNFDE (1.1) has a solution defined on $\mathcal{R}_+$.

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