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LAPLACE, INVERSE LAPLACE TRANSFORMS AND THEIR APPLICATIONS

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Abstract:

The Laplace transform is a powerful integral operator that converts complex functions from the time domain into simpler algebraic expressions in the frequency domain, facilitating the straightforward solution of ordinary and partial differential equations. Its inverse counterpart then returns the solution to its original context, enabling critical applications in engineering fields such as control systems analysis, electrical circuit design, signal processing, and mechanical vibration modeling.

Keywords: Laplace transforms, Inverse Laplace transforms, applications

PART 1: LAPLACE TRANSFORMS
1.1 Definition

The Laplace transform converts a time-domain function $f(t)$ into a complex frequency-domain function $F(s)$:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

where:

- t is time (real, $t \geq 0$)
- $s = \sigma + i\omega$ is a complex variable
- The integral converges for $\text{Re}(s) >$ (some constant)

1.2 Existence Conditions

The Laplace transform exists if:

1. $f(t)$ is piecewise continuous on $[0, \infty)$
2. $f(t)$ is of exponential order: $| f(t) | \leq Me^{at}$ for some M, a

1.3 Properties of Laplace Transforms

Property	Formula
Linearity	$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$
First Shift (Frequency)	$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
Second Shift (Time)	$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s)$

Property	Formula
Scaling	$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$
Time Derivative	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
Second Derivative	$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
n-th Derivative	$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
Time Integral	$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$
Convolution	$\mathcal{L}\{f * g\} = F(s)G(s)$
t-multiplication	$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^nF(s)}{ds^n}$
Division by t	$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$
Periodic Function	$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$
Initial Value	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
Final Value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

1.4 Common Laplace Transform Pairs

Time Domain $f(t)$	Laplace Domain $F(s)$	Conditions
$\delta(t)$ (Unit impulse)	1	-
$u(t)$ (Unit step)	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0, n = 0,1,2,\ldots$

Time Domain $f(t)$	Laplace Domain $F(s)$	Conditions
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	(s >k)
$\cosh(kt)$	$\frac{s}{s^2-k^2}$	(s >k)
$t\sin(kt)$	$\frac{2ks}{(s^2+k^2)^2}$	$s > 0$
$t\cos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$	$s > 0$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$	$s > a$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$	$s > a$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$	$a \neq b$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$s > 0$
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	$s > 0$
$\frac{e^{-a^2/4t}}{\sqrt{\pi t}}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$s > 0$

Time Domain $f(t)$	Laplace Domain $F(s)$	Conditions
$J_0(at)$ (Bessel)	$\frac{1}{\sqrt{s^2+a^2}}$	$s > 0$

1.5 Worked Examples - Laplace Transforms

Example 1: Find $\mathcal{L}\{e^{3t}\sin(2t)\}$

Solution:

From table: $\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$

Using first shift theorem with $a = 3$:

$$\mathcal{L}\{e^{3t}\sin(2t)\} = \frac{2}{(s-3)^2+4}$$

Example 2: Find $\mathcal{L}\{t^2\cos(t)\}$

Solution:

From table: $\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$

Using t-multiplication property ($n = 2$):

$$\mathcal{L}\{t^2\cos(t)\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

First derivative: $\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$

Second derivative: $\frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right) = \frac{-2s(s^2+1)^2-(1-s^2)\cdot 2(s^2+1)(2s)}{(s^2+1)^4}$

Simplifying: $\mathcal{L}\{t^2\cos(t)\} = \frac{2s(3-s^2)}{(s^2+1)^3}$

Example 3: Find $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2 \end{cases}$

Solution:

Write as: $f(t) = t + (2-t)u(t-2)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{(2-t)u(t-2)\}$$

For the second term, let $g(t) = 2-(t+2) = -t$, so $(2-t)u(t-2) = g(t-2)u(t-2)$ with $g(t) = -t$

Then $\mathcal{L}\{g(t-2)u(t-2)\} = e^{-2s}G(s) = e^{-2s}\mathcal{L}\{-t\} = -e^{-2s}\cdot\frac{1}{s^2}$

Therefore:

$$F(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

PART 2: INVERSE LAPLACE TRANSFORMS

2.1 Definition

The inverse Laplace transform recovers the time-domain function from its frequency-domain representation:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,F(s)\,ds$$

In practice, we use algebraic methods and tables rather than this complex integral.

2.2 Methods for Inverse Laplace Transforms

Method 1: Table Lookup

Match $F(s)$ with known transform pairs (see table in Section 1.4).

Method 2: Partial Fraction Decomposition

Case 1: Distinct Real Roots

$$F(s) = \frac{2s+1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Find A and B, then $f(t) = Ae^{-t} + Be^{-2t}$

Case 2: Repeated Real Roots

$$F(s) = \frac{3s+4}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$

Then $f(t) = Ae^{-2t} + Bte^{-2t} + \frac{C}{2}t^2e^{-2t}$

Case 3: Complex Roots

$$F(s) = \frac{2s+5}{s^2+4s+13} = \frac{2s+5}{(s+2)^2+9}$$

Rewrite numerator: $2(s+2)+1$, then:

$$f(t) = 2e^{-2t}\cos(3t) + \frac{1}{3}e^{-2t}\sin(3t)$$

Method 3: Completing the Square

For quadratics not in standard form:

$$F(s) = \frac{4s+7}{s^2+6s+25} = \frac{4s+7}{(s+3)^2+16}$$
$$= \frac{4(s+3)-5}{(s+3)^2+16} = 4\cdot\frac{s+3}{(s+3)^2+16} - \frac{5}{4}\cdot\frac{4}{(s+3)^2+16}$$
$$f(t) = 4e^{-3t}\cos(4t) - \frac{5}{4}e^{-3t}\sin(4t)$$

Method 4: Using Properties

First Shift Theorem:
If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

Second Shift Theorem:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

Convolution Theorem:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)d\tau$$

Method 5: Heaviside Expansion Formula

For $F(s) = \frac{P(s)}{Q(s)}$ with distinct roots $a_1, a_2, \ldots, a_n$:

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(a_k)}{Q'(a_k)} e^{a_k t}$$

2.3 Systematic Procedure for Inverse Laplace Transforms

1. **Simplify** $F(s)$ if possible
2. **Check for standard forms** in the table
3. **Perform partial fraction decomposition** for rational functions
4. **Complete the square** for quadratic denominators
5. **Apply shift theorems** to match table entries
6. **Use convolution** for products
7. **Combine results** to get $f(t)$

2.4 Worked Examples - Inverse Laplace Transforms

Example 1: Find $\mathcal{L}^{-1}\left\{\frac{3s+5}{s^2+2s+5}\right\}$

Solution:
Complete square: $s^2 + 2s + 5 = (s+1)^2 + 4$
Rewrite numerator: $3s + 5 = 3(s+1) + 2$

$$F(s) = \frac{3(s+1)}{(s+1)^2+4} + \frac{2}{(s+1)^2+4}$$
$$f(t) = 3e^{-t}\cos(2t) + e^{-t}\sin(2t)$$

Example 2: Find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$

Solution:
Use partial fractions:

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

By symmetry and comparing coefficients:

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}$$

$$f(t) = \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)$$

Example 3: Find $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\}$

Solution:

First, $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$

Using second shift theorem with $a = 3$:

$$f(t) = (t-3)e^{-(t-3)}u(t-3)$$

Example 4: Find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$

Solution:

Recall derivative property: $\mathcal{L}\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}$

For $k = 2$: $\mathcal{L}\{t\sin(2t)\} = \frac{4s}{(s^2+4)^2}$

Therefore: $\frac{s}{(s^2+4)^2} = \frac{1}{4} \cdot \frac{4s}{(s^2+4)^2}$

So: $f(t) = \frac{1}{4}t\sin(2t)$

PART 3: APPLICATIONS

3.1 Solving Ordinary Differential Equations (ODEs)

The most powerful application: converts ODEs to algebraic equations.

General Procedure:

1. Take Laplace transform of both sides
2. Substitute initial conditions
3. Solve for $Y(s)$
4. Find inverse Laplace transform to get $y(t)$

Example 3.1.1: First-Order ODE

Solve $y' + 2y = e^{-t}$, $y(0) = 1$

Step 1: Take Laplace transform

$$[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}$$

Step 2: Substitute $y(0) = 1$

$$sY(s) - 1 + 2Y(s) = \frac{1}{s+1}$$

Step 3: Solve for $Y(s)$

$$(s+2)Y(s) = 1 + \frac{1}{s+1}$$

$$Y(s) = \frac{1}{s+2} + \frac{1}{(s+1)(s+2)}$$

Step 4: Partial fractions for second term

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

So: $Y(s) = \frac{1}{s+2} + \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{s+1}$

Step 5: Inverse transform

$$y(t) = e^{-t}$$

Example 3.1.2: Second-Order ODE

Solve $y'' + 3y' + 2y = \sin(t), y(0) = 0, y'(0) = 1$

Step 1: Take Laplace transform

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s^2+1}$$

Step 2: Substitute initial conditions

$$s^2Y(s) - 1 + 3sY(s) + 2Y(s) = \frac{1}{s^2+1}$$

Step 3: Solve for $Y(s)$

$$(s^2 + 3s + 2)Y(s) = 1 + \frac{1}{s^2+1}$$
$$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)(s^2+1)}$$

Step 4: Partial fractions (detailed work omitted for brevity)

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{1/2}{s+1} - \frac{2/5}{s+2} + \frac{3/10}{s^2+1} - \frac{1/5}{s^2+1}$$

Step 5: Inverse transform

$$y(t) = \frac{3}{2}e^{-t} - \frac{7}{5}e^{-2t} + \frac{3}{10}\cos(t) - \frac{1}{5}\sin(t)$$

3.2 Electrical Circuit Analysis

Laplace transforms simplify circuit analysis by converting differential equations to algebraic equations using impedances.

Impedances in s-domain:

- Resistor: $Z_R = R$
- Inductor: $Z_L = sL$ (with initial current $i_L(0^-)$)
- Capacitor: $Z_C = \frac{1}{sC}$ (with initial voltage $v_C(0^-)$)

Example 3.2.1: RLC Circuit

Consider a series RLC circuit with $R = 2\Omega, L = 1H, C = 0.5F$, and voltage source $v(t) = 10u(t)$. Find current $i(t)$ assuming zero initial conditions.

Step 1: Write KVL equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t)$$

Step 2: Take Laplace transform

$$LsI(s) + RI(s) + \frac{1}{Cs}I(s) = V(s)$$

Step 3: Substitute values

$$sI(s) + 2I(s) + \frac{2}{s}I(s) = \frac{10}{s}$$

Step 4: Solve for $I(s)$

$$I(s)\left(s+2+\frac{2}{s}\right) = \frac{10}{s}$$

$$I(s)\left(\frac{s^2 + 2s + 2}{s}\right) = \frac{10}{s}$$

$$I(s) = \frac{10}{s^2 + 2s + 2}$$

Step 5: Complete square and inverse transform

$$I(s) = \frac{10}{(s+1)^2+1}$$

$$i(t) = 10e^{-t}\sin(t)$$

Example 3.2.2: Circuit with Initial Conditions

For the same circuit but with initial capacitor voltage $v_C(0^-) = 5V$, find $i(t)$.

Step 1: Include initial condition in transform

$$LsI(s) + RI(s) + \frac{1}{Cs}I(s) + \frac{v_C(0^-)}{s} = V(s)$$

Step 2: Substitute values

$$sI(s) + 2I(s) + \frac{2}{s}I(s) + \frac{5}{s} = \frac{10}{s}$$

Step 3: Solve for $I(s)$

$$I(s)\left(\frac{s^2 + 2s + 2}{s}\right) = \frac{10 - 5}{s} = \frac{5}{s}$$

$$I(s) = \frac{5}{s^2 + 2s + 2} = \frac{5}{(s+1)^2+1}$$

Step 4: Inverse transform

$$i(t) = 5e^{-t}\sin(t)$$

3.3 Control Systems Engineering

Laplace transforms are fundamental in control theory for analyzing system behavior.

Key Concepts:

- **Transfer Function**: $H(s) = \frac{Y(s)}{U(s)}$ (output/input with zero initial conditions)
- **Poles and Zeros**: Roots of denominator and numerator
- **Stability**: System stable if all poles have negative real parts
- **Step Response**: $Y(s) = H(s) \cdot \frac{1}{s}$

Example 3.3.1: Transfer Function and Step Response

A system has transfer function $H(s) = \frac{4}{s^2+2s+4}$. Find and analyze its step response.

Step 1: Step input $u(t)$ gives $U(s) = \frac{1}{s}$

$$Y(s) = H(s)U(s) = \frac{4}{s(s^2 + 2s + 4)}$$

Step 2: Partial fractions

$$\frac{4}{s(s^2 + 2s + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 4}$$

Solving: $A = 1, B = -1, C = -2$

Step 3: Complete square: $s^2 + 2s + 4 = (s + 1)^2 + 3$

$$Y(s) = \frac{1}{s} - \frac{s + 1}{(s+1)^2+3} - \frac{1}{(s+1)^2+3}$$

Step 4: Inverse transform

$$y(t) = 1 - e^{-t}\cos(\sqrt{3}t) - \frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)$$

Step 5: Analyze response

- Steady-state value: $\lim_{t\to\infty} y(t) = 1$
- Damping ratio: $\zeta = \frac{1}{\sqrt{3}} \approx 0.577$ (underdamped)
- Natural frequency: $\omega_n = 2$ rad/s

Example 3.3.2: System Stability

Determine stability of systems with transfer functions:

a) $H(s) = \frac{5}{s^2+3s+2}$
b) $H(s) = \frac{2s}{s^2-2s+5}$
c) $H(s) = \frac{s+1}{s^2+4}$

Solution:

a) Poles: $s = -1, -2 \to$ both negative real $\to$ **stable**

b) Poles: $s = 1 \pm 2i \to$ real part positive $\to$ **unstable**

c) Poles: $s = \pm 2i \to$ purely imaginary $\to$ **marginally stable**

3.4 Mechanical Systems

Example 3.4.1: Mass-Spring-Damper System

A mass $m = 2kg$ is attached to a spring with $k = 8N/m$ and damper with $c = 4Ns/m$. Find displacement $x(t)$ when force $f(t) = 10u(t)$ is applied, with initial conditions $x(0) = 0.1m$, $\dot{x}(0) = 0$.

Step 1: Equation of motion

$$m\ddot{x} + c\dot{x} + kx = f(t)$$
$$2\ddot{x} + 4\dot{x} + 8x = 10u(t)$$

Step 2: Take Laplace transform

$$2[s^2X(s) - sx(0) - \dot{x}(0)] + 4[sX(s) - x(0)] + 8X(s) = \frac{10}{s}$$

Step 3: Substitute initial conditions

$$2[s^2X(s) - 0.1s] + 4[sX(s) - 0.1] + 8X(s) = \frac{10}{s}$$
$$(2s^2 + 4s + 8)X(s) - 0.2s - 0.4 = \frac{10}{s}$$

Step 4: Solve for $X(s)$

$$(2s^2 + 4s + 8)X(s) = \frac{10}{s} + 0.2s + 0.4$$
$$X(s) = \frac{10/s + 0.2s + 0.4}{2(s^2 + 2s + 4)}$$

Step 5: Simplify and find inverse transform

$$X(s) = \frac{5}{s(s^2 + 2s + 4)} + \frac{0.1s + 0.2}{s^2 + 2s + 4}$$

Complete square: $s^2 + 2s + 4 = (s + 1)^2 + 3$

After partial fractions and inverse transform:

$$x(t) = 1.25 - e^{-t}[1.15\cos(\sqrt{3}t) + 0.23\sin(\sqrt{3}t)]$$

3.5 Signal Processing

Example 3.5.1: Convolution and System Response

A system has impulse response $h(t) = e^{-2t}u(t)$. Find output when input $x(t) = e^{-t}u(t)$.

Method 1: Time-domain convolution

$$y(t) = \int_0^t e^{-2\tau} \cdot e^{-(t-\tau)} d\tau = e^{-t} \int_0^t e^{-\tau}\, d\tau = e^{-t}(1 - e^{-t})$$

Method 2: Laplace transform

$$H(s) = \frac{1}{s+2}, X(s) = \frac{1}{s+1}$$
$$Y(s) = H(s)X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$
$$y(t) = e^{-t} - e^{-2t}$$

Note: Both methods give the same result!

Example 3.5.2: Filter Analysis

A low-pass filter has transfer function $H(s) = \frac{1}{s+1}$. Find:

a) Impulse response
b) Step response
c) Response to $x(t) = \sin(2t)$

Solution:

a) $h(t) = \mathcal{L}^{-1}\{\frac{1}{s+1}\} = e^{-t}u(t)$

b) Step response:

$$Y(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$
$$y(t) = (1 - e^{-t})u(t)$$

c) Response to $\sin(2t)$:

$$X(s) = \frac{2}{s^2+4}$$
$$Y(s) = \frac{2}{(s+1)(s^2+4)} = \frac{2/5}{s+1} - \frac{2s/5}{s^2+4} + \frac{2/5}{s^2+4}$$
$$y(t) = \frac{2}{5}e^{-t} - \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t)$$

3.6 Partial Differential Equations (PDEs)

Laplace transforms can solve PDEs by transforming the time variable, reducing the PDE to an ODE.

Example 3.6.1: Heat Equation

Solve the heat equation for a semi-infinite bar:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, x > 0, t > 0$$

with boundary conditions: $u(0,t) = T_0$ (constant), $u(x,t)$ bounded as $x \to \infty$
initial condition: $u(x,0) = 0$

Step 1: Take Laplace transform with respect to t

$$sU(x,s) - u(x,0) = k\frac{\partial^2 U}{\partial x^2}$$

Since $u(x,0) = 0$: $sU = k\frac{d^2U}{dx^2}$

Step 2: This is an ODE in x:

$$\frac{d^2U}{dx^2} - \frac{s}{k}U = 0$$

Step 3: General solution:

$$U(x,s) = A(s)e^{-\sqrt{s/k}\,x} + B(s)e^{\sqrt{s/k}\,x}$$

Boundedness as $x \to \infty$ requires $B(s) = 0$

Step 4: Apply boundary condition at $x = 0$:

$$U(0,s) = \mathcal{L}\{u(0,t)\} = \mathcal{L}\{T_0\} = \frac{T_0}{s}$$

Thus $A(s) = \frac{T_0}{s}$

Step 5:

$$U(x,s) = \frac{T_0}{s}e^{-\sqrt{s/k}\,x}$$

Step 6: Inverse Laplace transform (using table)

$$u(x,t) = T_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$$

where erfc is the complementary error function.

3.7 Applications in Biology and Chemistry

Example 3.7.1: Pharmacokinetics - Drug Concentration

A drug is administered intravenously at time $t = 0$. The concentration in the bloodstream follows:

$$\frac{dC}{dt} = -kC, C(0) = C_0$$

Solution: Taking Laplace transform:

$$sC(s) - C_0 = -kC(s)$$
$$C(s) = \frac{C_0}{s+k}$$
$$C(t) = C_0e^{-kt}$$

Example 3.7.2: Two-Compartment Model

A more realistic model with central and peripheral compartments:

$$\frac{dC_1}{dt} = -(k_{10} + k_{12})C_1 + k_{21}C_2 + \frac{D}{V_1}\delta(t)$$
$$\frac{dC_2}{dt} = k_{12}C_1 - k_{21}C_2$$

Taking Laplace transforms and solving gives biexponential decay:

$$C_1(t) = Ae^{-\alpha t} + Be^{-\beta t}$$

3.8 Economics and Finance

Example 3.8.1: Continuous Compound Interest

If money grows at continuous rate r, with initial investment P:

$$\frac{dA}{dt} = rA, A(0) = P$$

Solution: $A(s) = \frac{P}{s-r}$, so $A(t) = Pe^{rt}$

Example 3.8.2: Loan Repayment

For a loan with continuous repayment at rate m:

$$\frac{dA}{dt} = rA - m, A(0) = L \text{ (initial loan)}$$

Solution:

$$A(s) = \frac{L}{s-r} - \frac{m}{s(s-r)}$$
$$A(t) = Le^{rt} - \frac{m}{r}(e^{rt} - 1)$$

3.9 Chemical Reaction Kinetics

Example 3.9.1: First-Order Reaction Series

For consecutive first-order reactions: $A \overset{k_1}{\rightarrow} B \overset{k_2}{\rightarrow} C$

Equations:

$$\frac{d[A]}{dt} = -k_1[A], [A](0) = A_0$$
$$\frac{d[B]}{dt} = k_1[A] - k_2[B], [B](0) = 0$$
$$\frac{d[C]}{dt} = k_2[B], [C](0) = 0$$

Solution using Laplace transforms:

$$[A](s) = \frac{A_0}{s+k_1} \Rightarrow [A](t) = A_0 e^{-k_1 t}$$
$$[B](s) = \frac{k_1 A_0}{(s+k_1)(s+k_2)} = \frac{k_1 A_0}{k_2 - k_1}\left(\frac{1}{s+k_1} - \frac{1}{s+k_2}\right)$$
$$[B](t) = \frac{k_1 A_0}{k_2 - k_1}(e^{-k_1 t} - e^{-k_2 t})$$
$$[C](t) = A_0 - [A](t) - [B](t)$$

PART 4: SUMMARY AND COMPARISON

5.1 When to Use Laplace Transforms

Application	Why Laplace Transform?
ODEs with constant coefficients	Converts to algebraic equations
Circuits with capacitors/inductors	Impedance concept simplifies analysis
Control systems	Transfer functions reveal stability/response
Systems with discontinuous inputs	Handles step/impulse functions naturally
Convolution integrals	Becomes multiplication in s-domain
Initial value problems	Initial conditions automatically included

5.2 Advantages and Limitations

Advantages:

☑ Converts differential equations to algebraic equations
☑ Automatically incorporates initial conditions
☑ Handles discontinuous functions easily
☑ Provides frequency-domain insight
☑ Enables transfer function approach
☑ Simplifies convolution to multiplication

Limitations:

✕ Requires $t \geq 0$ (not suitable for negative time)
✕ Inverse transform can be difficult
✕ Not suitable for nonlinear equations
✕ Some functions have no Laplace transform
✕ Numerical inversion can be unstable

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