JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

A ψ - FUNCTION-BASED DECOMPOSITION OF THE MULTIVARIABLE I- FUNCTION.

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Abstract : In this paper, we establish a new expansion formula for the multivariable I – function by employing the generalized ψ - function introduced by Srivastava, Gupta, and Goyal. Using the classical identity of Nair [6] together with the series representation of the generalized ψ - function, we derive a decomposition formula expressing the multivariable I – function as an infinite series.

***IndexTerms* - Multivariable I- function, Gamma function, Pochhammer symbol, Hypergeometric function, Generalized ψ - function.**

1. INTRODUCTION

Notations and Results used :

$(a)_n$ stands for $a(a+1)...(a+n-1)$.

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, n \geq 1 \tag{1.1}$$

${}_1(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)};A_j)_p$ stands for $(a_1;\alpha_1^{(1)},...,\alpha_1^{(r)};A_1),(a_2;\alpha_2^{(1)},...,\alpha_2^{(r)};A_2),...,(a_p;\alpha_p^{(1)},...,\alpha_p^{(r)};A_p)$.

The generalized Fox's H-function, namely the *I*-function of *r*-variables introduced by Prathima, Nambisan and Santha Kumari [7, p.38] is defined and represented as:

$$I[z_1,\ldots,z_r] = I^{0,n:\ m_1,n_1;\ldots;m_r,n_r}_{p,q:\ p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)};A_j)_p : {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};...;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},...,\beta_j^{(r)};B_j)_Q : {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};...;{}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{Lr}\theta_1(s_1)...\theta_r(s_r)\ \phi(s_1,...,s_r)\ z_1^{s_1}...z_r^{s_r}ds_1...ds_r, \tag{1.2}$$

where $\phi(s_1,\ldots,s_r)$ and $\theta_i(s_i),\ i=1,2,...,r$ are given by,

$$\phi(s_1,...,s_r) = \frac{\prod_{j=1}^{n}\Gamma^{A_j}(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}{\prod_{j=1}^{q}\Gamma^{B_j}(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i)\prod_{j=n+1}^{p}\Gamma^{A_j}(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}, \tag{1.3}$$

$$\theta_i(s_i)=\frac{\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}. \tag{1.4}$$

Also $z_i\neq 0\ (i=1,...,r),\ \omega=\sqrt{-1},\ m_j,n_j,p_j,q_j\ (j=1,...,r)\ ,n\ ,p\ ,q$ are non-negative integers such that $0\le n\le p,\quad q\ge 0,\quad 0\le m_j\le q_j,\quad 0\le n_j\le p_j\quad (j=1,2....,r)$ (not all zero simultaneously). $\alpha_j^{(i)}\ (j=1,2,...,p\ ,\ i=1,2...,r)$, $\beta_j^{(i)}\ (j=1,2,...,q,\ \ i=1,2,...,r)$, $\gamma_j^{(i)}\ (j=1,2,...,p_i,\ i=1,2,...,r)$, and $\delta_j^{(i)}\ (j=1,2,...,q_i,\ i=1,2,...,r)$ are positive numbers. $a_j\ (j=1,2,...,p)$, $b_j\ (j=1,2,...,q)$, $c_j^{(i)}\ (j=1,2,...,p_i,\ i=1,2,...,r)$, and $d_j^{(i)}\ (j=1,2,...,q_i,\ i=1,2,...,r)$ are complex numbers. The exponents $A_j\ (j=1,2,...,p)$, $B_j\ (j=1,2,...,q)$, $C_j^{(i)}\ (j=1,2,...,p_i,\ i=1,2,...,r)$, and $D_j^{(i)}\ (j=1,2,...,q_i,\ i=1,2,...,r)$ of various gamma functions may take non-integer values. The I-function of r-variables is analytic if

$$\Psi_i=\sum_{j=1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\ \le 0, i=1,2,...,r.$$

The integral (1.2) converges absolutely if $\left|\arg(z_i)\right|<\frac{1}{2}\Delta_i\pi\ ,\ i=1,2,\ldots,r$ where

$$\Delta_i=-\sum_{j=n+1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)}+\sum_{j=1}^{n_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\gamma_j^{i}>0\quad . \tag{1.5}$$

Remark: 1

If $D_j^{(i)}=1\ \ (j=1,2,\ldots m_i,\ \ i=1,2,\ldots r)$ in (1.2), then the I-function will be denoted by:

$$\overline{I}[z_1,......,z_r]=I^{0,n\,:\,m_1,n_1\,;...;m_r,n_r}_{p,q\,:\,p_1,q_1\,;...;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}I_1\\ \\ I_2\end{matrix}\right]=\frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r}\overline{\theta}_1(s_1)\ldots\overline{\theta}_r(s_r)\phi(s_1,...,s_r)\ z_1^{s_1}\ldots z_r^{s_r}\ ds_1\ldots ds_r \tag{1.6}$$

where

$$I_1={}_1(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)};\ A_j)_p\ :\ {}_1(c_j^{(1)},\gamma_j^{(1)};\ C_j^{(1)})_{p_1};...;{}_1(c_j^{(r)},\gamma_j^{(r)};\ C_j^{(r)})_{p_r},$$

$$I_2={}_1(b_j;\beta_j^{(1)},...,\beta_j^{(r)};\ B_j)_q\ :\ {}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1},{}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};...;{}_1(d_j^{(r)},\delta_j^{(r)};1)_{m_r},{}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r},$$

$$\overline{\theta}_i(s_i)\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\ \prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\ \prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)},\ i=1,2,\ldots,r \tag{1.7}$$

and $\phi(s_1,...,s_r)$ is given by (1.3).

The integral (1.6) converges absolutely if $\left|\arg(z_i)\right|<\frac{1}{2}\Delta_i'\,\pi,\ i=1,2,\ldots,r.$

where,

$$\Delta_i'=\left(-\sum_{j=n+1}^{p}A_j\,\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\,\beta_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}\,\delta_j^{(i)}+\sum_{j=1}^{n_i}C_j^{(i)}\,\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\,\gamma_j^{(i)}\right)>0,\ i=1,2,...,r. \tag{1.8}$$

Remark: 2

If $C_j^{(i)} = 1 \ (j = 1,2,\ldots n_i)$, $D_j^{(i)} = 1 \ (j = 1,2,\ldots m_i)$ for $i = 1,2,\ldots r$ and if $n = 0$ in (1.2), then the corresponding function will be denoted by:

$$\overline{I_1}[z_1,\ldots\ldots,z_r] = I_{p,q\,:\,p_1,q_1\,;\ldots;\,p_r,q_r}^{0,0\,:\,m_1,n_1\,;\ldots;\,m_r,n_r}\left[\begin{array}{c|c} z_1 & I_1 \\ \vdots & \\ z_r & I_2 \end{array}\right] = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\overline{\theta_1}(s_1)\ldots\overline{\theta_r}(s_r)\,\phi_1(s_1,\ldots,s_r)\; z_1^{s_1}\ldots z_r^{s_r}\; ds_1\ldots ds_r \tag{1.9}$$

where

$$I_1 = {}_1(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};\,A_j)_p : {}_1(c_j^{(1)},\gamma_j^{(1)};1)_{n_1},\,{}_{n_1+1}(c_j^{(1)},\gamma_j^{(1)};\,C_j^{(1)})_{p_1};\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};1)_{n_r},\,{}_{n_r+1}(c_j^{(r)},\gamma_j^{(r)};\,C_j^{(r)})_{p_r}, \tag{1.10}$$

$$I_2 = {}_1(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};\,B_j)_q : {}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1},\,{}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ldots;{}_1(d_j^{(r)},\delta_j^{(r)};1)_{m_r},\,{}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}, \tag{1.11}$$

$$\phi_1(s_1,\ldots,s_r) = \frac{1}{\prod_{j=1}^{q}\Gamma^{B_j}(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i)\,\prod_{j=1}^{p}\Gamma^{A_j}(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}, \tag{1.12}$$

$$\overline{\theta_i}(s_i)\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\,\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\,\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)},\qquad i = 1,2,\ldots,r\ . \tag{1.13}$$

The integral (1.9) converges absolutely if $|\arg(z_i)| < \frac{1}{2}\Delta_i''\pi,\ i = 1,2,\ldots,r.$

where,

$$\Delta_i'' = \left(-\sum_{j=1}^{p}A_j\,\alpha_j^{(i)} - \sum_{j=1}^{q}B_j\,\beta_j^{(i)} + \sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}D_j^{(i)}\,\delta_j^{(i)} + \sum_{j=1}^{n_i}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\,\gamma_j^{(i)}\right) > 0,\ i = 1,2,\ldots,r. \tag{1.14}$$

For further details refer [7].

Nair [6, p.234]

$$\alpha^s\prod_{J=1}^{l}\frac{\Gamma(d_j-\delta_j s)}{\Gamma(c_j-\delta_j s)} = \sum_{k=0}^{\infty}\frac{(s)_k}{k!\,\Gamma(-k)}\;{}_{l+1}\psi_l\left[\begin{array}{c}(-k,1),\ {}_1(d_j,\delta_j)_l \\ {}_1(c_j,\delta_j)_l\end{array};\tfrac{1}{\alpha}\right] \tag{1.15}$$

provided $\alpha > 1$ and $\mathrm{Re}(c_j) > \mathrm{Re}(d_j) > 0,\ \ j = 1,2,\ldots,l$. and

$$\alpha^s\prod_{J=1}^{l}\frac{\Gamma(d_j+\delta_j s)}{\Gamma(c_j+\delta_j s)} = \sum_{k=0}^{\infty}\frac{(-1/\alpha)^k\,(s)_k}{k!\,\Gamma(-k)}\;{}_{l+1}\psi_l\left[\begin{array}{c}(-k,1),\ {}_1(d_j,\delta_j)_l \\ {}_1(c_j,\delta_j)_l\end{array};\alpha\right] \tag{1.16}$$

provided $\frac{1}{2} < \alpha < 1$ and $\mathrm{Re}(c_j) > \mathrm{Re}(d_j) > 0,\ \ j = 1,2,\ldots,l$.

$$(-1)^k\,(s_1)_k = \frac{\Gamma(1-s_1)}{\Gamma(1-s_1-k)} \tag{1.17}$$

H.M Srivastava, K.C. Gupta, S.P. Goyal [12, p.2]
Generalized Hypergeometric function

$${}_pF_q\left[\begin{array}{c}(a_p); \\ (b_q);\end{array} z\right] = {}_pF_q\left[(a_p);(b_q);z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!} \tag{1.18}$$

The series on the right-hand side of (1.18) is absolutely convergent for all values of z, real or complex, when $p \le q$. Also, when $p = q+1$, the series is convergent if $|z|<1$.

It converges when $z=1$, if $\mathrm{Re}\left[\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right] > 0$ and when $z=-1$, if $\mathrm{Re}\left[\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right] > -1$.

If $p > q+1$, the series never converges except when $z=0$, and the function is only defined when the series terminates.

H.M Srivastava, K.C. Gupta, S.P. Goyal [12, p.19]

$$ {}_p\psi_q \left[\begin{matrix} {}_1(a_j,\alpha_j)_p; \\ {}_1(b_j,\beta_j)_q; \end{matrix} z \right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j r)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j r)} \frac{z^r}{r!} \tag{1.19} $$

When $\alpha_j = 1,\ j=1,2,\ldots,p$ and $\beta_j = 1,\ j=1,2,\ldots,q$, (1.19) becomes:

$$ {}_p\psi_q \left[\begin{matrix} {}_1(a_j,1)_p; \\ {}_1(b_j,1)_q; \end{matrix} z \right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + r)}{\prod_{j=1}^{q} \Gamma(b_j + r)} \frac{z^r}{r!} = \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j}\, {}_pF_q \left[\begin{matrix} (a)_p; \\ (b)_q; \end{matrix} z \right] \tag{1.20} $$

H.M Srivastava, K.C. Gupta, S.P. Goyal [12, p.18]

$$ {}_1F_0 \left[\begin{matrix} a; \\ -; \end{matrix} -z \right] = (1+z)^{-a} \tag{1.21} $$

2. Main Result:

$$ I^{0,0:\, m_1,n_1+l\ ;\ldots;m_r,n_r}_{p,q:\, p_1+l,q_1+l;\ldots;p_r,q_r} \left[\begin{matrix} z_1\alpha \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} I_1 \\ \\ I_2 \end{matrix} \right] $$

$$ = \sum_{k=0}^{\infty} \frac{1}{k!\alpha^k\,\Gamma(-k)}\, {}_{l+1}\psi_l \left[\begin{matrix} (-k,1),\ {}_1\left(1-c_j,\delta_j\right)_l \\ {}_1\left(1-d_j,\delta_j\right)_l \end{matrix} ; \alpha \right] I^{0,0:\, m_1+1,\, n_1\ ;\ldots;m_r,n_r}_{p,q:p_1+1,q_1+1;\ldots;p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} I_3 \\ \\ I_4 \end{matrix} \right], \text{ when } \tfrac{1}{2} < \alpha < 1 \tag{2.1} $$

$$ = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(-k)}\, {}_{l+1}\psi_l \left[\begin{matrix} (-k,1),\ {}_1\left(1-c_j,\delta_j\right)_l \\ {}_1\left(1-d_j,\delta_j\right)_l \end{matrix} ; \alpha \right] I^{0,0:\, m_1+1,\, n_1\ ;\ldots;m_r,n_r}_{p,q:p_1+1,q_1+1;\ldots;p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} I_5 \\ \\ I_6 \end{matrix} \right], \text{ when } 0 < \alpha < 1 \tag{2.2} $$

where

$$ I_1 = {}_1\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_p : {}_1\left(c_j,\delta_j;1\right)_l, {}_1\left(c_j^{(1)},\gamma_j^{(1)};1\right)_{n_1}, {}_{n_1+1}\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1}; \ldots; {}_1\left(c_j^{(r)},\gamma_j^{(r)};1\right)_{n_r}, {}_{n_r+1}\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r}, \tag{2.3} $$

$$ I_2 = {}_1\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_q : {}_1\left(d_j^{(1)},\delta_j^{(1)};1\right)_{m_1}, {}_{m_1+1}\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1}, {}_1\left(d_j,\delta_j;1\right)_l; {}_1\left(d_j^{(2)},\delta_j^{(2)};1\right)_{m_2}, {}_{m_2+1}\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2};\ldots $$
$$ \ldots; {}_1\left(d_j^{(r)},\delta_j^{(r)};1\right)_{m_r}, {}_{m_r+1}\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r}, \tag{2.4} $$

$$ I_3 = {}_1\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_p : {}_1\left(c_j^{(1)},\gamma_j^{(1)};1\right)_{n_1}, {}_{n_1+1}\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1}, (1-k,1;1); $$
$$ {}_1\left(c_j^{(2)},\gamma_j^{(2)};1\right)_{n_2}, {}_{n_2+1}\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_2}; \ldots; {}_1\left(c_j^{(r)},\gamma_j^{(r)};1\right)_{n_r}, {}_{n_r+1}\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r}, \tag{2.5} $$

$$ I_4 = {}_1\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_q : (1,1;1)\, {}_1\left(d_j^{(1)},\delta_j^{(1)};1\right)_{m_1}, {}_{m_1+1}\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1};\ldots; {}_1\left(d_j^{(r)},\delta_j^{(r)};1\right)_{m_r}, {}_{m_r+1}\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r} \tag{2.6} $$

$$ I_5 = {}_1\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_p : {}_1\left(c_j^{(1)},\gamma_j^{(1)};1\right)_{n_1}, {}_{n_1+1}\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1}, (0,1;1); $$
$$ {}_1\left(c_j^{(2)},\gamma_j^{(2)};1\right)_{n_2}, {}_{n_2+1}\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_2}; \ldots; {}_1\left(c_j^{(r)},\gamma_j^{(r)};1\right)_{n_r}, {}_{n_r+1}\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r}, \tag{2.7} $$

$$ I_6 = {}_1\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_q : (k,1;1)\, {}_1\left(d_j^{(1)},\delta_j^{(1)};1\right)_{m_1}, {}_{m_1+1}\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1};\ldots; {}_1\left(d_j^{(r)},\delta_j^{(r)};1\right)_{m_r}, {}_{m_r+1}\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r}. \tag{2.8} $$

provided

(i) $\mathrm{Re}(c_j) > \mathrm{Re}(d_j) > 0,\ \ j=1,2,\ldots,l$, (2.9)

(ii) $\Delta_i'' > 0, i=1,2,\ldots,r$, (2.10)

(iii) $|\arg(z_i)| < \frac{1}{2}\Delta_i''\pi, i = 1,2,...,r$ (2.11)

where Δ_i'' is given by (1.14).

Proof:

Express the Left-hand side of (2.1) in terms of contour integral using (1.9), it becomes:

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{Lr}\overline{\theta}_1(s_1)...\overline{\theta}_r(s_r)\ \phi_1(s_1,...,s_r)\ (z_1\alpha)^{s_1} z_2^{s_2}...z_r^{s_r}\prod_{j=1}^{l}\frac{\Gamma\left(1-c_j+\delta_j s_1\right)}{\Gamma\left(1-d_j+\delta_j s_1\right)}ds_1...ds_r \tag{2.12}$$

where $\phi_1(s_1,...,s_r)$ and $\overline{\theta_i}(s_i),\quad i=1,...,r$ are given by (1.12) and (1.13) respectively.

Using (1.16), the expression (2.12) reduces to:

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{Lr}\theta_1(s_1)...\theta_r(s_r)\ \phi_1\ (s_1,...,s_r)\ z_1^{s_1}...z_r^{s_r}\sum_{k=0}^{\infty}\frac{(-1/\alpha)^k (s_1)_k}{k!\Gamma(-k)}\ {}_{l+1}\psi_l\left[\begin{matrix}(-k,1),\ {}_1(1-c_j,\delta_j)_l\\ {}_1(1-d_j,\delta_j)_l\end{matrix};\alpha\right]ds_1...ds_r \tag{2.13}$$

Interchanging the order of integration and summation, which is justified under the given conditions, and using (1.17), (2.13) reduces to:

$$\sum_{k=0}^{\infty}\frac{1}{k!\alpha^k\Gamma(-k)}\ {}_{l+1}\psi_l\left[\begin{matrix}(-k,1),\ {}_1(1-c_j,\delta_j)_l\\ {}_1(1-d_j,\delta_j)_l\end{matrix};\alpha\right]\frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{Lr}\theta_1(s_1)...\theta_r(s_r)\ \phi_1\ (s_1,...,s_r)\ \frac{\Gamma(1-s_1)}{\Gamma(1-s_1-k)}z_1^{s_1}...z_r^{s_r}\ ds_1...ds_r \tag{2.14}$$

Now interpret these integrals in (2.14) with the help of (1.9) to get the right-hand side of (2.1).

The change in the order of summation and integrations which are justified because the series $\sum_{k=0}^{\infty}\frac{(-1)^k (s_1)_k}{k!\alpha^k\Gamma(-k)}\ {}_{l+1}\psi_l\left[\begin{matrix}(-k,1),\ {}_1(1-c_j,\delta_j)_l\\ {}_1(1-d_j,\delta_j)_l\end{matrix};\alpha\right]$ converges uniformly in any arbitrary interval $a \le s_1 \le b$,

$\theta_1(s_1)...\theta_r(s_r)\phi_1\ (s_1,...,s_r)z_1^{s_1}...z_r^{s_r}$ are continuous and the integrals

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{Lr}\theta_1(s_1)...\theta_r(s_r)\ \phi_1(s_1,...,s_r)\ z_1^{s_1}...z_r^{s_r}\sum_{k=0}^{\infty}\frac{(-1)^k (s_1)_k}{k!\alpha^k\Gamma(-k)}\ {}_{l+1}\psi_l\left[\begin{matrix}(-k,1),\ {}_1(1-c_j,\delta_j)_l\\ {}_1(1-d_j,\delta_j)_l\end{matrix};\alpha\right]ds_1...ds_r$$

exist, when the given conditions are satisfied.

The proof of (2.2) is similar; instead of (1.16), (1.15) is used with α and $s_1,...,s_r$ are replaced by $\frac{1}{\alpha}$ and $-s_1,...,-s_r$ respectively.

3. Special Cases:

When $\delta_j = 1$, for $j = 1,2,...,l$, and using (1.20), (2.1) and (2.2) become:

$$I^{0,0:\ m_1,n_1+l\ ;...;m_r,n_r}_{p,q:p_1+l,q_1+l;...;p_r,q_r}\left[\begin{matrix}z_1\alpha\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}J_1\\ \\ J_2\end{matrix}\right]$$

$$=\prod_{j=1}^{l}\frac{\Gamma(1-c_j)}{\Gamma(1-d_j)}\sum_{k=0}^{\infty}\frac{1}{k!\alpha^k}\ {}_{l+1}F_l\left[\begin{matrix}-k,\ {}_1(1-c_j)_l\\ {}_1(1-d_j)_l\end{matrix};\alpha\right]I^{0,0:\ m_1+1,\ n_1\ ;...;m_r,n_r}_{p,q:p_1+1,q_1+1;...;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}I_3\\ \\ I_4\end{matrix}\right],\ \text{when } \tfrac{1}{2}<\alpha<1 \tag{3.1}$$

$$=\prod_{j=1}^{l}\frac{\Gamma(1-c_j)}{\Gamma(1-d_j)}\sum_{k=0}^{\infty}\frac{1}{k!}\ {}_{l+1}F_l\left[\begin{matrix}-k,\ {}_1(1-c_j)_l\\ {}_1(1-d_j)_l\end{matrix};\alpha\right]I^{0,0:\ m_1+1,\ n_1\ ;...;m_r,n_r}_{p,q:p_1+1,q_1+1;...;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}I_5\\ \\ I_6\end{matrix}\right],\ \text{when } 0<\alpha<1 \tag{3.2}$$

where

$$J_1 = {}_1\left(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)};A_j\right)_p : {}_1\left(c_j,1;1\right)_l , {}_1\left(c_j^{(1)},\gamma_j^{(1)};1\right)_{n_1} , {}_{n_1+1}\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1} ; \ldots ; {}_1\left(c_j^{(r)},\gamma_j^{(r)};1\right)_{n_r} , {}_{n_r+1}\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r} ,$$

$$J_2 = {}_1\left(b_j;\beta_j^{(1)},...,\beta_j^{(r)};B_j\right)_q : {}_1\left(d_j^{(1)},\delta_j^{(1)};1\right)_{m_1} , {}_{m_1+1}\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1} , {}_1\left(d_j,1;1\right)_l ; {}_1\left(d_j^{(2)},\delta_j^{(2)};1\right)_{m_2} , {}_{m_2+1}\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2} ;\ldots$$
$$\ldots ; {}_1\left(d_j^{(r)},\delta_j^{(r)};1\right)_{m_r} , {}_{m_r+1}\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r} ,$$

and I_3, I_4, I_5, I_6 are given by (2.5), (2.6), (2.7) and (2.8) respectively.

provided the conditions (2.9), (2.10) and (2.11) are satisfied with $\delta_j = 1$, for $j = 1, 2, \ldots, l$.

When $l = 0$, the conditions on α can be relaxed and using (1.21), the results (3.1) and (3.2) take the following form.

$$I^{0,0:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1\alpha \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} K_1 \\ \\ K_2 \end{matrix}\right]$$

$$= \sum_{k=0}^{\infty} \frac{\left(1/\alpha - 1\right)^k}{k!} I^{0,0:m_1+1,\ n_1\ ;\ldots;m_r,n_r}_{p,q:p_1+1,q_1+1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} I_3 \\ \\ I_4 \end{matrix}\right], \text{ provided } \left|1/\alpha - 1\right| < 1 \qquad (3.3)$$

$$= \sum_{k=0}^{\infty} \frac{\left(1-\alpha\right)^k}{k!} I^{0,0:m_1+1,\ n_1\ ;\ldots;m_r,n_r}_{p,q:p_1+1,q_1+1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} I_5 \\ \\ I_6 \end{matrix}\right], \text{ provided } \left|1-\alpha\right| < 1 \qquad (3.4)$$

where

$$K_1 = {}_1\left(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)};A_j\right)_p : {}_1\left(c_j^{(1)},\gamma_j^{(1)};1\right)_{n_1} , {}_{n_1+1}\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1} ;\ldots; {}_1\left(c_j^{(r)},\gamma_j^{(r)};1\right)_{n_r} , {}_{n_r+1}\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r} ,$$

$$K_2 = {}_1\left(b_j;\beta_j^{(1)},...,\beta_j^{(r)};B_j\right)_q : {}_1\left(d_j^{(1)},\delta_j^{(1)};1\right)_{m_1} , {}_{m_1+1}\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1} ;\ldots\ldots; {}_1\left(d_j^{(r)},\delta_j^{(r)};1\right)_{m_r} , {}_{m_r+1}\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r} ,$$

and I_3, I_4, I_5, I_6 are given by (2.5), (2.6), (2.7) and (2.8) respectively.

The additional conditions in the results (3.3) and (3.4) are $\Delta_i'' > 0$ and $\left|\arg(z_i)\right| < \frac{1}{2}\Delta_i''\pi,\ i = 1, 2, \ldots, r.$

Putting $p = q = 0,\ r = 1,\quad C_j^{(i)} = 1,\ n_i + 1 \le j \le p_i,\ i = 1, 2, \ldots, r,\ D_j^{(i)} = 1,\ \ m_i + 1 \le j \le q_i,\ i = 1, 2, \ldots, r$ and specialising the parameters in (2.1), (2.2), (3.1), (3.2), (3.3) and (3.4), the results established by Nair V.C [6] are obtained.

On account of the most general characters of the result (2.1), due to the presence of I – functions of several variables, many special cases of known and unknown can be derived.

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