



Fixed Point Results in Metric Spaces Using Set-based Iteration Functions

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Abstract

In this paper, we establish several unique fixed point (UFP) results in a Real Metric (RM) space using a novel set-based iteration framework. Set-based iteration function provide a natural and robust extension of classical iteration by allowing mapping from element to subsets of the underlying space. The approach is based on the construction of an ordered pair **G-set** and its associated **G₀-subset**, generated through iterative sequences of a self-mapping. Unlike classical pointwise iterations, the proposed framework interprets the iteration process as a sequence of subsets governed by metric distances. We prove the existence and uniqueness of fixed-points for Banach, Kannan, Chatterjea, and generalized contractive mappings by analyzing the monotonic behaviour of an associated distance T- sequence. The completeness of the metric space ensures convergence, while the contraction conditions guarantee uniqueness. The results unify and extend several classical fixed point theorems and provide a flexible structure applicable to nonlinear analysis, optimization, coding theory, and control systems. The exposition is self-contained and written to meet contemporary research standards.

Keywords

Set-based iteration, unique fixed point, G-set, G₀-subset, metric space, contraction mapping, monotone sequence

1. Introduction

Fixed point theory is a cornerstone of nonlinear analysis and plays a fundamental role in diverse fields such as optimization, computer science, control theory, DNA coding, fuzzy systems, and communication engineering. Since Banach's celebrated contraction principle [2], numerous generalizations have been developed to handle broader classes of mappings and spaces. Recent studies emphasize graph-based and set-based structures in metric spaces, which provide richer frameworks for analyzing convergence behavior [3, 6, 7, 8]. Motivated by these developments, we introduce a set-based iteration approach using ordered pairs, called the G-set, and its associated G₀-subset generated by iterative mappings. The fixed point theorem was extended by the work of Kannan [9] in the year 1968. The multivalued contraction mapping result on unique fixed point theorems in

complete metric spaces was introduced introduction Nadaler [10,11]. The iterated functions consist of F-contraction fixed point theory as extended by Secelean [12] in the year 2013. Some fixed point results via R-functions are based on the work done by Nastasi, and

Vetro[13]. A Common Fixed Point (CFP) theorem in metric space is presented in the work of Sedge,Shobe and Zhou [14]. The fixed point theorem is generalized in metric space and mapping induced by Dhage [15] in the year 1992. The fixed point theorems on weakly contractive maps are derived from the work done by Rhoades [16].The fixed point theory of contemporary mathematics has extended the work of Reich[17].An adaptive fuzzy logic design controller by using a genetic algorithm. International Proceedings Conference (IFC) on genetic algorithms by Karr [18] in 1991.

The novelty of this work lies in

- Constructing fixed-point iterations as ordered pair sets
- Associating a distance-based monotone sequence (T-sequence)
- Establishing unique fixed point results under various contractive conditions

This approach generalizes classical Banach, Kannan and Chatterjea contractions within a unified framework.

2. Preliminaries

Let (M^*, d) be a Real Metric (RM) space. Denote by the family of all nonempty subsets of M^* .

Definition 2.1 (Contraction Mapping)

A mapping $F: M^* \rightarrow M^*$ is called a **Banach contraction** if there exists $\alpha \in [0,1)$

Such that, $d(Fk, Fz) \leq \alpha d(k, z), \forall k, z \in M^*$

Definition 2.2 (Kannan and Chatterjea Mappings)

A mapping $F: M^* \rightarrow M^*$ is

- **Kannan mapping** if

$$d(Fk, Fz) \leq \alpha(d(k, Fk) + d(z, Fz)), \alpha \in (0,1)$$

- **Chatterjea mapping** if

$$d(Fk, Fz) \leq \alpha(d(k, Fz) + d(z, Fk)), \alpha \in (0,1)$$

Definition 2.3 (Completeness)

(M^*, d) is complete if every Cauchy sequence in M^* converges to a point of M^* .

Definition 2.4 Ordered Metric Spaces

Combining order relation with set-based iterations.

3. Sequences

3.1 Let us (M^*, d) be a Real Metric (RM) space, and a sequence $\{k\}_{n=1}^{\infty}$ in the real metric (M^*, d) is said to be convergent if for every $\varepsilon > 0$, \exists, N_I , such that $d(k_n, k) < \varepsilon, n \geq N$

3.2 Let us (M^*, d) be a Real Metric (RM) space and a sequence $\{k\}_{n=1}^{\infty}$ is said to be monotonic if either $k_n < k_{n+1}, n \in I$ (non-decreasing). If a sequence $\{k\}_{n=1}^{\infty}$ is monotonic and bounded then the sequence $\{k\}_{n=1}^{\infty}$ is convergent.

3.3 Let us (M^*, d) be a Real Metric (RM) space, and a sequence $\{k\}_{n=1}^{\infty}$ is said to be Cauchy if for a given $\varepsilon > 0$, $\exists, N_2 \in I$, such that, $d(k_n, k_m), n, m \geq N_2$

4. Set-Based Iteration Framework

Let G and G_0 be two sets. If an ordered pair G -set $= (V, E)$, where V and E are two real points of G -set, V points the vertical line and other points E are on the horizontal line. Again we let G_0 is a subset of G -set and $G_0 = (V_0, E_0)$ where real sub-sequences $V_0 \in V$ and other sub-sequence $E_0 \in E$. We assume that (M^*, d) is a real metric defined on G -set. Let a function $F: M^* \rightarrow M^*$, then we now define the order pair G -set associated with the real metric.

Such that,

$$\begin{aligned} z_1 &= F^1 z, \text{ where } z_0 \in M^*, \\ z_2 &= Fz_1 = F(Fz_0) = F^2 z_0 \\ z_3 &= F(Fz_2) = F(F^2 z_0) = F^3 z_0 \\ z_n &= F^n z_0 \end{aligned}$$

z_n iterates of z and $z_0, Fz_0, F^2 z_0, F^3 z_0 \dots F^n z_0$ are the iteration function in real number sequence of F . Now we have G_0 -set associated by iteration function with the metric space such that,

$$\begin{aligned} V_0 &= \{ z_0, Fz_0, F^2 z_0, F^3 z_0, F^4 z_0, F^5 z_0, \dots, F^{n-1} z_0, F^n z_0 \} \\ E_0 &= \{ d(z_0, Fz_0), d(Fz_0, F^2 z_0), d(F^2 z_0, F^3 z_0), d(F^3 z_0, F^4 z_0), \dots, d(F^{n-1} z_0, F^n z_0) \} \end{aligned}$$

These metric spaces are made by using iterated real number sequences with G -set.

Let $F: M^* \rightarrow M^*$ and $z_0 \in M^*, z_0 \in M^*$. Define the iteration: $z_n = F^n z_0, n \in \mathbb{N}$.

Definition 4.1 (G-set)

The **ordered pair G-set** associated with F is defined as

$$G = (V, E), G = (V, E), G = (V, E),$$

Where $V = M^*, E = \{(k, Fk): k \in M^*\}$.

Definition 4.2 (Go-subset)

The **Go-subset** induced by z_0 is $G_0 = (V_0, E_0)$,

Where, $V_0 = \{z_0, Fz_0, F^2 z_0, \dots\}, E_0 = \{d(z_0, Fz_0), d(Fz_0, F^2 z_0), \dots\}$.

Definition 4.3 (T-sequence)

Define, $T_n = d(F^{n-1}z_0, F^n z_0)$.

The sequence $\{T_n\}$ is called the **T-sequence** associated with the G_0 -subset.

5. Auxiliary Results

Lemma 5.1

If the T-sequence $\{T_n\}$ is non-increasing. (M^*, d) is a Real Metric (RM) space and again Let $F: M^* \rightarrow M^*$ is a mapping in which the ordered pair G-set is associated with M^* . Let z_0 is any arbitrary point of M^* . Assume G_0 -set is the subset of G-set defined as in definition [4.2]. Then the iterated monotonically increasing sequence $\{z_0, Fz_0, F^2z_0, \dots, F^n z_0\}$ is Cauchy .

Proof

Let T-sequence associated with the G_0 -set is non-increasing. Such as, $T_1 > T_2 > T_3 > T_4 > \dots > T_n > T_{n+1} > \dots$. Therefore, the T-sequence of the length of the sequences $\{z_0, Fz_0, F^2z_0, \dots\}$ is a sequence of non-negative numbers bounded below by zero. By definition [3.1], the T-sequence converges . This implies that the terms of the sequences $\{z_0, Fz_0, F^2z_0, \dots\}$ are getting closer and approaching approach to infinity. The iterated monotonically increasing sequences $\{z_0, Fz_0, F^2z_0, \dots\}$ form a Cauchy. Since $\{T_n\}$ is monotone and bounded below by 0, it converges. For any $m > n$, $d(F^n z_0, F^m z_0) \leq \sum_{k=1}^{m-n} T_k$. Hence, the sequence is Cauchy.

Lemma 5.2

Let (M^*, d) is a Real Metric (RM) space. Let a function $F: M^* \rightarrow M^*$ and again let an ordered pair G-set associated with M^* . Then the point z^* of M^* is a Fixed Point (FP) of F if the G-set has a loop at z^* .

Proof

Let z^* is the fixed point of F . Then $F(z^*) = z^*$. According to the definition of [4.1], the ordered pair G-set $(F(z^*), z^*) \in G$. Then that is (z^*, z^*) , $b \in G$ implies G has a loop at z^* .

A point $z^* \in M^*$, $M^* z^* \in M^*$ is a fixed point of F if and only if the G-set has a loop at z^* .

6. Main Results

Theorem 6.1 (Banach-Type Unique Fixed Point Theorem)

Statement

Let (M^*, d) be a Complete Real Metric (CRM) space and let $F: M^* \rightarrow M^*$ be a contraction mapping, i.e., there exists $\alpha \in [0, 1)$ such that $d(Fk, Fz) \leq \alpha d(k, z), \forall k, z \in M^*$ Then F has a unique fixed point in M^*

Proof

Let $z_0 \in M^*$ be arbitrary and define the iterative sequence $z_n = F^n z_0$, $n \in \mathbb{N}$

Define the associated **T-sequence**, $T_n = d(z_{n-1}, z_n) = d(F^{n-1} z_0, F^n z_0)$

Using the contraction property,

$$T_{n+1} = d(z_n, z_{n+1}) = d(Fz_{n-1}, Fz_n) \leq \alpha d(z_{n-1}, z_n)$$

$$= \alpha T_n \quad T_{n+1} = d(z_n, z_{n+1}) = d(Fz_{n-1}, Fz_n)$$

$$d(z_n, z_{n+1}) = d(Fz_{n-1}, Fz_n) \leq \alpha d(z_{n-1}, z_n) = \alpha T_n. \text{ Since } 0 \leq \alpha < 1$$

We have $T_{n+1} < T_n$, hence $\{T_n\}$ is **strictly decreasing** and bounded below by 0.

Therefore, $\{T_n\}$ converges. By **Lemma 5.1**, the iteration sequence $\{z_n\}$ is a **Cauchy sequence**.

Since (M^*, d) is complete, there exists $z^* \in M^*$ such that $z_n \rightarrow z^*$

Now, using continuity of F

$$\begin{aligned} Fz^* &= F(\lim_{n \rightarrow \infty} z_n) = \lim_{n \rightarrow \infty} Fz_n = \lim_{n \rightarrow \infty} z_{n+1} \\ &= z^* \quad Fz^* \\ &= F(\lim_{n \rightarrow \infty} z_n) = \lim_{n \rightarrow \infty} Fz_n = \lim_{n \rightarrow \infty} z_{n+1} = z^* \end{aligned}$$

Hence, z^* is a **fixed point** of F .

Uniqueness

Assume c^* is another fixed point. Then $d(z^*, c^*) = d(Fz^*, Fc^*) \leq \alpha d(z^*, c^*)$. Since $0 \leq \alpha < 1$

This implies, $d(z^*, c^*) = 0$, hence $z^* = c^*$ $z^* = c^*$

Therefore, F has a **unique fixed point** in M^*

Theorem 6.2 (Kannan-Type Unique Fixed Point Theorem)**Statement**

Let (M^*, d) be a CRM space and $F: M^* \rightarrow M^*$ satisfy,

$$d(Fk, Fz) \leq \alpha [d(k, Fk) + d(z, Fz)] \dots [1] \quad \forall k, z \in M^*, \text{ where } \alpha \in (0, 1)$$

Then F has a **unique fixed point** in M^*

Proof

Let $z_0 \in M^*$ $z_0, M^* z_0 \in M^*$ and define $z_n = F^n z_0$. Define $T_n = d(z_{n-1}, z_n)$

Applying the given condition with $k = z_{n-1}$ and $z = z_n$,

$$\begin{aligned} T_{n+1} &= d(z_n, z_{n+1}) \leq \alpha [d(z_{n-1}, z_n) + d(z_n, z_{n+1})] \\ &= \alpha (T_n + T_{n+1}) \Rightarrow T_{n+1} \leq \alpha T_n \end{aligned}$$

Rearranging, $(1-\alpha)T_{n+1} \leq \alpha T_n \Rightarrow T_{n+1} \leq \frac{\alpha}{1-\alpha} T_n$

$$T_n (1-\alpha) T_{n+1} \leq \alpha T_n \Rightarrow T_{n+1} \leq \frac{\alpha}{1-\alpha} T_n$$

Since $\alpha < 1$, hence $\{T_n\}$ is decreasing and convergent. By **Lemma 5.1**, $\{z_n\}$ is Cauchy. Completeness of M^* implies convergence to some $z^* \in M^*$ and $M^* z^* \in M^*$

Using the contractive inequality and $\lim_{n \rightarrow \infty}$

$$d(z^*, Fz^*) = 0, \text{ So, } Fz^* = z^*$$

Uniqueness

If possible c^* is another fixed point of F , then $Fc^* = c^*$

From equation [1]

$$d(Fz^*, Fz^*) \leq \alpha [d(z^*, Fz^*) + d(z^*, Fz^*)] = 0$$

Where $0 < \alpha < 1$

$$d(z^*, c^*) \leq \alpha [d(z^*, z^*) + d(c^*, c^*)] = 0, \Rightarrow d(z^*, c^*) = 0$$

Hence, $z^* = c^* = z^* = c^*$

Thus, F has a **unique fixed point**.

Theorem 6.3 (Chatterjea-Type Unique Fixed Point Theorem)

Statement

Let (M^*, d) be a CRM space and $F: M^* \rightarrow M^*$ satisfy. $d(Fk, Fz) \leq \alpha [d(k, Fz) + d(z, Fk)]$

$\forall k, z \in M^*$, where $\alpha \in (0, 1)$. Then F has a unique fixed point.

Proof

Let $z_n = F^n z_0$ and define $T_n = d(z_{n-1}, z_n)$

Using the condition with $k = z_{n-1}$, $z = z_n$,

$$T_{n+1} \leq \alpha [d(z_{n-1}, z_{n+1}) + d(z_n, z_n)] = \alpha d(z_{n-1}, z_{n+1})$$

By triangle inequality, $d(z_{n-1}, z_{n+1}) \leq T_n + T_{n+1}$

Hence, $T_{n+1} \leq \alpha(T_n + T_{n+1})$. Which yields, $T_{n+1} \leq \alpha_1 - \alpha T_n < T_n$

Thus, $\{T_n\}$ is decreasing and convergent, and $\{z_n\}$ is Cauchy. Completeness gives convergence to z^* . Passing to the limit in the inequality gives $Fz^* = z^*$. Uniqueness follows exactly as in Theorem 6.2. Hence, F admits a unique fixed point.

Theorem 6.4 (Generalized Contractive Mapping)

Statement

Let (M^*, d) be a CRM space and $F: M^* \rightarrow M^*$ satisfy.

$$d(Fk, Fz) \leq \alpha_1 [d(k, Fk) + d(z, Fz)] + \alpha_2 [d(k, Fz) + d(z, Fk)],$$

where $0 < \alpha_1, \alpha_2 < 1$ and $\alpha_1 + \alpha_2 < 1$. Then F has a unique fixed point.

Proof

Let $z_n = F^n z_0$, $T_n = d(z_{n-1}, z_n)$.

Applying the inequality, $T_{n+1} \leq (\alpha_1 + \alpha_2)(T_n + T_{n+1})$.

Thus, $(1 - \alpha_1 - \alpha_2)T_{n+1} \leq (\alpha_1 + \alpha_2)T_n$,

$T_{n+1} \leq \beta T_n$, where $\beta = 1 - \alpha_1 - \alpha_2$, $\alpha_1 + \alpha_2 < 1$.

Hence $\{T_n\}$ is decreasing and convergent. By Lemma 5.1, $\{z_n\}$ is Cauchy and converges to z^* .

$\lim_{n \rightarrow \infty} T_n = 0$ in the contractive inequality yields, $d(z^*, Fz^*) = 0 \Rightarrow Fz^* = z^*$.

Uniqueness

Assume z^* and c^* are two fixed points. Then $d(z^*, c^*) \leq 2\alpha_2 d(z^*, c^*)$. Since $2\alpha_2 < 1$, it follows that $d(z^*, c^*) = 0$, hence $z^* = c^*$. Therefore, F has a **unique fixed point** in M^* .

Conclusion and Future Work

This paper introduces a set-based iteration methodology for fixed point analysis in metric spaces. By associating iterative mappings with ordered pair G -sets and monotone T -sequences, we establish several unique fixed point results that generalize classical contraction principles.

Future directions include

- Extension to **fuzzy metric spaces**
- Applications in **machine learning convergence**
- Analysis in **fractal geometry and hybrid dynamical systems**

Author Contributions

¹Dr. Raja Ram Singh-40%, ²Saurabh Kumar-15%, ³Laxman Kumar- 15%

⁴Sanjay Kumar- 15%, ⁵ Ravi Kumar Nirdosh- 15%

Use of AI tools declaration

The author declare the have not used AI tools in the creation of this paper.

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