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The B-K's Fixed Point Theorem in a CCRM-Spaces

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Abstract :In this paper, we obtain the B –K's (Banach –Kannan's)fixed point theorem result in a CCRM (Complete Cone Rectangular Metric) –Spaces. These results are generalizations of some of the well known results existing in the literature.

IndexTerms–Cone metric space , contractive condition, fixed point, normal cone.
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I. INTRODUTION AND PRELIMINARIES

The concept of cone metric space introduced by Huang and Zhang [4] , where the set of real numbers is replaced by an ordered Banach space and obtained some fixed point results in cone metric space. Later on several Mathematicians have been working in these results, and they extended these results in different ways (see for e.g. [1-3, -12]). Branciari [ ] , Azam , Arshad and Beg [2] extended cone metric space into cone rectangular metric space. Recently, Jelli and Samet [5] obtained the Kanna's fixed point theorem in a cone rectangular metric space. In this paper we obtain, the B-K's (Banach –Kannan's) fixed point theorem in a CCRM(Complete Cone Rectangular Metric) -spaces.
The following are needed to obain our main result which are due to [4, 5].

Definition .1.1. Let M be always a real Banach space and Q is a subset of M, Q is called a cone if and only if :

(i) Q is closed , non empty and Q $\neq \{0\}$,

(ii) $\alpha x + \beta y \in Q$ for all x, y ϵ Q and α, βϵR,

(iii) xϵQ and -xϵQ implies x = 0.

Definition 1.2. Given a cone $Q \subset M$, we define a partial ordering $\leq$ on M with respect to Q by x << y if x < y and x ≠ y. we shall write x << y if y-x ϵ interior of Q . The cone Q is called normal if there is a number L > 0 such that for all x, y ϵL,

$$0 \leq x \leq y \text{ implies } \| x\| \leq L\|y\|.$$

The least positive number L satisfying the above is called the normal constant of Q.

In the following we always suppose M is a Banach space , Q is a cone in M with int P ≠∅ and ≤ is partial ordering with respect to P.

Definition 1.3. Let X be a non- empty set . Suppose that the mapping $\rho : X \times X \to M$ satisfies the following

(a) ρ(x, y) > 0 for all x, y ϵ X, x ≠ y, and ρ(x, y) = 0 if and only if x = y;

(b) ρ(x, y) = ρ(y, x) , for all x, y ϵ X;

(c) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$,

for all x, y, w ,zϵ X and for all distinct points w,z ϵX \{x, y} (rectangular property). Then ρ is called a cone rectangular metric on X and (X, ρ) is called a cone rectangular metric space.

Note that any cone metric space is a a cone rectangular metric space but the converse is not true in general.

Example 1.4. Let E = R^2 , P = { (x, y) ϵM / x. y ≥ 0}, ρ:X × X → M such that P(x, y) = { (0 , 0) if x = y ; (3α, 3) if x , y are in {1.2}, x ≠y; (α, 1) if x and y can not both at a time in {1. 2}, x ≠ y, where α > 0 is a constant . Then (X, ρ) is a cone rectangular metric space but it is not a cone metric space since we have ρ(1, 2) = (3α, 3) > ρ(1, 3) + ρ(3,2) = (2α, 2).

Lemma 1.5. Let (X ,ρ) be a cone rectangular metric space , Q be a normal cone. Let (x_n) be a sequence in X. Then $x_n \to x$ as $n\to\infty$ iff $\|\rho(x_n, x)\| \to 0$ as $n\to\infty$.

Definition 1.6. Let (X, ρ) be a cone rectangular metric space. Let $\{x_n\}$ be a sequence in X and x ϵ X. If for every c ϵM with 0<<M there is an $n_0 \epsilon N$ such that $\rho(x_n , x) << c$, then $\{x_n\}$ is said to be convergence , $\{x_n\}$ convergences to x and x is the limit point of $\{x_n\}$. We denote this by $x_n \to x$, as $n \to\infty$.

Definition 1.7 Let (X, ρ) be a cone rectangular metric space. Let $\{x_n\}$ be a sequence in X and x ϵ X. If for every c ϵM with 0<< M there is an m,nϵN such that $\rho(x_n , x_m) << c$, then $\{x_n\}$ is said to be a Cauchy sequence.

Lemma 1.8 . Let (X, ρ) be a cone rectangular metric space and Q be a normal cone . Let $\{x_n\}$ be a sequence in X and x ϵ X. then $\{x_n\}$ is a Cauchy sequence if and only if $\rho(x_n , x_m) \to 0$ as n, m→∞.

Definition 1.8 Let (X, ρ) be a cone rectangular metric space. If every Cauchy sequence is convergent in X, then X is called a complete one rectangular metric space.

2. Main Result

Theorem 2.1. Let (X, ρ) be a CCRM-space. Let P be a normal cone with normal constant H. Suppose a mapping A; X→X satisfies the following

$$\rho(Ax, Ay) \le a\, \rho(x, y) + b[\rho(Ax, x) + \rho(Ay, y)]$$

then (i) A has a unique fixed point in X.

(ii) For every xϵX the iterative sequence $(A^n x)$ converges to the fixed point.

Proof: Let xϵ X . We have

$$\rho(Ax , A^2 x) \le a\, \rho(x, Ax) +b[\rho(Ax ,x) + \rho(AAx , Ax)]$$
$$\le a\, \rho(x, Ax) +b[\rho(Ax ,x) + \rho(Ax , A^2 x)](1-b)\, \rho(Ax , A^2x)$$
$$\le (a+b)\, \rho(Ax , x)$$
$$\le (a+b)\ /(1-b)\ \ \rho(Ax ,x)$$

Again $\rho(A^2 x , A^3 x) \le a\, \rho(Ax, A^2 x) +b[\rho(Ax ,A^2 x) + \rho(A^2 x , A^3 x)]$

$\rho(A^2 x , A^3 x) \le a+b /1-b\ \ \rho(Ax, A^2 x)$

$\le (a+b /1-b)^2\ \ \rho(x, A x).$

Thus in general if n is positive integer then

$\rho(A^2 x , A^3 x) \le (a+b /1-b)^n\ \ \rho(x, A x)$

$\le k^n\ \rho(x, A x)$, where k = (a+b /1-b)ϵ [0,1).

We divide the proof into two cases .

Case-I : Let $A^m x = A^n x$ for some m,nϵN, m ≠n. Let m > n .

Then $A^{m-n}(A^n x) = A^n x$. That is, $A^q y = y$, where q = m-n, $y = A^n x$.

Now since q > 1 where$P(y, Ay) = \rho(A^q x, A^{q+1}) \le k^n\, \rho(y, Ay)$

Since k ϵ [0, 1) we obtain

- ρ (y, Ay) ϵ P and ρ (y, Ay) ϵ P,

Which implies that ‖ ρ (y, Ay)‖ = 0. That is, Ay = y.

Case-II: Assume that $A^m x \ne A^n x$, for all m,n ϵN, m ≠ n. Clearly we have

$\rho(A^n x,\ A^{n+1}x) \le k^n\, \rho(x, Ax) \le k^n\ / 1- k\ \rho(x, Ax),$

and $\rho(A^n x,\ A^{n+2}x) \le a\, \rho(A^{n-1} x, A^{n+1} x) +b[\rho(A^{n-1} x ,A^n x) + \rho(A^{n+1} x , A^{n+2} x)]$

$$\le a\, [\rho(A^{n-1} x, A^n x)+ \rho(A^n x, A^{n+1} x)\ \ ] +b[\rho(A^{n-1} x ,A^n x) + \rho(A^{n+1} x , A^{n+2} x)],$$
$$\le (a+b)\ \rho(A^{n-1} x, A^n x) + a\, \rho(A^n x, A^{n+1} x) + b\, \rho(A^{n+1} x , A^{n+2} x),$$
$$\le (a+b)\, k^{n-1}\ \ \rho(x, A x) + a\, k^n\ \rho(x, A x) + b\, k^{n+1}\ \rho(x , A x),$$
$$\le (a+b)\, k^n / k\ \ \rho(x, A x) + a\, k^n\ \rho(x, A x) + b\, k^{n+1}\ \rho(x , A x),$$
$$\le (a+b)\, k^n / k\ \ \rho(x, A x).$$

If m > 2 is odd then written m = 2l + 1, l ≥ 1 and using the fact that $A^q x \ne A^k x$ for q,kϵN, q ≠k, we can easily see that

$$\rho(A^n x,\ A^{n+m}x) \le \rho(A^n x, A^{n+1} x) + \rho(A^{n+1} x ,A^{n+2} x) +\ldots + \rho(A^{n+2l} x , A^{n+2l+1} x),$$
$$\le k^n\, \rho(x, Ax) + k^{n+1}\ \rho(x ,A x) +\ldots + k^{n+2}\, \rho(x , A x),$$
$$\le k^n/1-k\ \ \rho(x, Ax).$$

Again if m > 2 is even the writing m = 2l , l ≥ 2 and using the same arguments as before, we can get that

$$\rho(A^n x,\ A^{n+m}x) \le \rho(A^n x, A^{n+2} x) + \rho(A^{n+2} x ,A^{n+3} x) +\ldots + \rho(A^{n+2l-1} x , A^{n+2l} x),$$
$$\le k^n\, \rho(x, Ax) + k^{n+2}\ \rho(x ,A x) +\ldots + k^{n+2l-1}\, \rho(x , A x),$$
$$\le k^n/1-k\ \ \rho(x, Ax).$$

Thus combining all the cases we have

$$\rho(A^n x,\ A^{n+m}x) \le k^n/1-k\ \ \rho(x, Ax) , \text{ for all } m,n\, \epsilon N.$$

Hence we get that

$$\| \rho(A^n x,\ A^{n+m}x) \| \le h\, k^n/1-k\ \ \|\rho(x, Ax)\| , \text{ for all } m, n\, \epsilon N.$$

Since, $h\, k^n/1-k\ \ \|\rho(x, Ax)\| \to 0$, as n → ∞.Therefore $\{A^n x\}$ is a Cauchy sequence. By the completeness of X there exists $x^* \epsilon X$ such that $A^n x \to x^*$, as n → ∞.

Now we shall show that $Ax^* = x^*$. Without loss of generality we can assume that $A^r x^* \ne x^*, A^r x^*$ for any rϵN, we have

$$\rho(x^*,\ A x^*) \le \rho(x^*,\ A^n x) + \rho(A^n x , A^{n+1}x\ ) + \rho(A^{n+1} x , Ax^*)$$
$$\le \rho(x^*,\ A^n x) + \rho(A^n x , A^{n+1}x\ ) + a\, \rho(A^n x , x^*) + b[\ \rho(A^n x , x^*) + \rho(x^* , Ax^*)]$$
$$\le \rho(x^*,\ A^n x) + \rho(A^n x , A^{n+1}x\ ) +(a + b)\, \rho(A^n x , x^*) + b\rho(x^* , Ax^*)$$
$$\rho(x^*,\ A x^*) \le 1+(a+b)\ \ \rho(A^n x , x^*) + \ \rho(A^n x , A^{n+1}x\ ) + b\rho(x^* , Ax^*)$$
$$\le 1+(a+b) / 1-b\ \ \rho(A^n x , x^*) + 1/ 1-b\ \rho(A^n x , A^{n+1}x)$$
$$\|\rho(x^*,\ A x^*)\| \le h\, [1+(a+b) / 1-b\ \ \|\rho(A^n x , x^*) \|+ 1/ 1-b\ \|\rho(A^n x , A^{n+1}x)\|] \to 0 \text{ as } n\to\infty.$$

Implies that $\rho(x^*,\ A x^*) = 0$. That is, $Ax^* = x^*$.

Uniqueness; If x_1^* is another fixed point of A then

$\rho(x^*, x_1^*) \le a\ \ \rho(Ax^*,\ A x_1^*) + b[\ \rho(\ x^* , Ax^*\ ) + \rho(\ x_1^* ,\ A x_1^*)]$

$\|\rho(x^*, x_1^*)\| \leq h\, a\, \|\rho(Ax^*, A x_1^*)\|$
$\|\rho(x^*, x_1^*)\| \leq 0,$
Implies that $\rho(x^*, x_1^*) = 0$, that is , $x^* = x_1^*$.
This completes the proof of the theorem.

REMARK. If we take a = 0 in the above theorem we get the theorem 2.1. of [5].

CONCLUSION Our results are extended and general results than the results of [5]

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