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Study of Rate of Convergence Radon Diffusion Equation of Fractional order in Soil Medium

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Abstract : In this paper, we develop implicit finite difference scheme for time fractional radon diffusion equation in soil medium. We discuss the stability and rate of convergence of the scheme. As an application of this scheme, we obtain the numerical solutions of the text problem and represented graphically by mathematical software Mathematica.

IndexTerms **- Fractional calculus, Implicit Finite difference, Caputo formula, Stability, Convergence.**

1. INTRODUCTION

Recent applications of fractional calculus (FC) focus on modeling systems with memory, hereditary properties, and complex dynamics, outperforming integer-order models in accuracy. Key areas include viscoelasticity, chaotic systems, image processing , bioengineering (bio-heat transfer), and control theory for autonomous vehicles [1,2,6]. In the development of fractional calculus and applications anomalous diffusion equation has received great interest. A physical approach to anomalous diffusion equation containing fractional order derivatives in time or space or time-space. As analytical solution of fractional diffusion equation is very difficult to find thus researchers develop the finite difference schemes to find numerical solution [3, 4, 7, 8, 9, 10, 11]. Radon is naturally occurring radioactive gas which is colorless, odorless and comes from the decay of uranium in rocks, soil and groundwater. Radon is present outdoors and indoors. Due to hazards properties of radon researchers have great interest to study the radon transport through soil, activated charcoal, concrete, etc. [5, 12, 13, 14, 15, 16]. In this paper we study the diffusion of radon in an activated charcoal medium. The diffusion theory came from the famous German physiologist Adolf Fick (1829-1901). He stated that the flux density J is proportional to the gradient of concentration. This gives,

$$\mathrm{J} = -D\frac{\partial \mathrm{C}}{\partial \mathrm{t}}$$

where J is the radon flux density is diffusion coefficient , $\frac{\partial \mathrm{C}}{\partial \mathrm{t}}$ is gradient of radon concentration and D is diffusivity coefficient of radon.

Now the change in concentration to change in time and position is stated by the Fick's second law which is the extension of Fick's first law, that gives,

$$\frac{\partial \mathrm{C(x,t)}}{\partial \mathrm{t}} = \frac{\partial^2 \mathrm{C(x,t)}}{\partial x^2} - \lambda \mathrm{c(x,t)}$$

where $\lambda = 2.1\times10^{-6}s^{-1}$ is the decay constant. A theoretical study of radon measurements with activated charcoal was studied by Nikezic and Urosevic [17]. In this study we develop the time fractional Implicit and Crank-Nicolson finite difference method for fractional order radon diffusion equation. We consider the following time fractional radon diffusion equation [TFRDE],

$$\frac{\partial^\alpha \mathrm{C(x,t)}}{\partial t^\alpha} = D\frac{\partial^2 \mathrm{C(x,t)}}{\partial t^2} - \lambda \mathrm{C(x,t)}, 0 < x < L, 0 \le \alpha \le 1, t \ge 0 \quad (1.1)$$

$$\textit{initial condition}: \ C(x,0) = 0, 0 < x < L \quad (1.2)$$

$$\textit{boundary conditions}: C(0,t) = c_0 \ \textit{and} \ \frac{\partial \mathrm{C(x,t)}}{\partial \mathrm{t}} = 0, t \ge 0 \quad (1.3)$$

Definition 1.1:- The Caputo time-fractional derivative of order $\alpha, (0 < \alpha \le 1)$ is defined by,

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial C(x,t)}{\partial \eta}\frac{d\eta}{(t-\eta)^\alpha}\ ;0<\alpha<1$$
$$=\frac{\partial C(x,t)}{\partial \eta}\ ; \qquad \alpha=1$$

2. FINITE DIFFERENCE SCHEME:

2.1 Implicit Finite Difference Scheme for TFRDE:

In this section, we develop the time fractional implicit finite difference method for fractional order radon diffusion equation (1.1)-(1.3).
We define,

$$t_k=k\tau\ ;\ k=0,1,2,\dots,N \ and\ x_i=ih\ ;\ i=0,1,2,\dots,N$$

where

$$\tau=\frac{T}{N}\ \ and\ h=\frac{L}{M}$$

Let $C(x_{i,}t_k); i=0,1,2,\dots,M\ and\ k=0,1,2,\dots,N$ be the exact solution of time fractional radon diffusion equation (TFRDE) (1.1)-(1.3) at mesh point $(x_{i,}t_k)$. Let C_i^k be the numerical approximation of the point $C(x_{i,}t_k)$. The time fractional derivative is approximated by the following scheme,

$$\frac{\partial^\alpha C(x_{i,}t_k)}{\partial t^\alpha}\approx\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{C(x_{i,}t_{j+1})-C(x_{i,}t_j)}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{d\eta}{(t_{k+1}-\eta)^\alpha}+O(\tau)$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{C(x_{i,}t_{j+1})-C(x_{i,}t_j)}{\tau}\int_{(k-j)\tau}^{(k-j+1)\tau}\frac{d\xi}{\xi^\alpha}+O(\tau)$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{C(x_{i,}t_{j+1})-C(x_{i,}t_j)}{\tau}\left[\frac{(j+1)^{1-\alpha}-j^{1-\alpha}}{1-\alpha}\right]\tau^{1-\alpha}\ +O(\tau)$$
$$=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[C_i^{k+1}-C_i^k\right]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j\left[C_i^{k-j+1}-C_i^{k-j}\right]+O(\tau)$$

where $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}, j=0,1,2,\dots,N$

Now for $\frac{\partial^2 C}{\partial x^2}$, we adopt a symmetric second order difference quotient in space at time level $t=t_{k+1}$

$$\frac{\partial^2 C}{\partial x^2}=\frac{C(x_{i-1,}t_{k+1})-2C(x_{i,}t_{k+1})+C(x_{i+1,}t_{k+1})}{h^2}$$

$$\frac{\partial^2 C}{\partial x^2}=\frac{C_{i-1}^{k+1}-C_i^{k+1}+C_{i+1}^{k+1}}{h^2}$$

Therefore, substituting in equation (1.1), we get

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[C_i^{k+1}-C_i^k\right]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j\left[C_i^{k-j+1}-C_i^{k-j}\right]=D\left[\frac{C_{i-1}^{k+1}-C_i^{k+1}+C_{i+1}^{k+1}}{h^2}\right]-\lambda C(x_{i,}t_k)$$

where $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}; j=0,1,2,\dots,k.$

Therefore, the complete fractional approximated initial boundary value problem is,

$$-rc_{i-1}^1+(1+2r)c_i^1-rc_{i+1}^1=(1-\mu)c_i^0;for\ k=0 \qquad (2.1)$$

$$-rc_{i-1}^{k+1}+(1+2r)c_i^{k+1}-rc_{i+1}^{k+1}=(1-\mu-b_1)c_i^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})c_i^{k-j}+b_kc_i^0\ ;$$
$$for\ k\geq 1 \qquad (2.2)$$

initial condition: $c_i^0, i=0,1,2,\dots,M$ (2.3)

boundary conditions: $c_0^k=c_0$ and $c_{M+1}^{k+1}=c_{M-1}^{k+1}.; k=0,1,2,\dots$ (2.4)

where $r=\dfrac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}$; $\mu=\ \lambda\Gamma(2-\alpha)\tau^\alpha; b_j=(j+1)^{1-\alpha}-j^{1-\alpha};\ j=0,1,2,\dots,k$;

$i=0,1,2,\dots,M\quad and\quad k=0,1,2,\dots,N$

The problem (2.1)-(2.4) is a complete discretization of the problem (1.1)-(1.3).

Therefore, the fractional approximated initial boundary value problem (2.1)-(2.4) can be written in the following matrix equation form

$$AC^1=(1-\mu)C^0+E;for\ k=0 \qquad (2.5)$$

$$AC^{k+1}=(1-\mu-b_1)C^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})C^{k-j}+b_kC^0+F\ ;$$
$$for\ k\geq 1 \qquad (2.6)$$

where

$$A = \begin{pmatrix} 1+2r & -r & \cdots & \cdots & \cdots \\ -r & 1+2r & -r & \cdots & \cdots \\ \vdots & \vdots & \ddots & -2r & 1+2r \end{pmatrix};$$

$C^1 = [c_1^1, c_2^1, c_3^1, \dots\dots\dots, c_M^1]^T; C^0 = [c_1^0, c_2^0, c_3^0, \dots\dots\dots, c_M^0]^T; C^k = [c_1^k, c_2^k, c_3^k, \dots\dots\dots, c_M^k]^T;$

$E = [rc_0^1, 0, \dots\dots\dots\dots, 0]^T; F = [rc_0^{k+1}, 0, \dots\dots\dots\dots, 0]^T; r = \dfrac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}$; $\mu = \lambda\Gamma(2-\alpha)\tau^\alpha$;

$b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$; $j = 0,1,2,\dots,k$; $i = 0,1,2,\dots,M$ *and* $k = 0,1,2,\dots,N$

3. STABILITY AND CONVERGENCE OF THE SCHEME:

3.1 Implicit Finite Difference Scheme for TFRDE:

Lemma 3.1.1:- [17] If $\lambda_j(A); j = 1,2,\dots,M-1$ represents eigenvalues of matrix A then we prove the following results,

1) $|\lambda_j(A)| \geq 1 ; j = 1,2,\dots,M-1$
2) $||A^{-1}||_2 \leq 1$

Theorem(3.1.1):-The solution of the fractional approximated IBVP (2.1)-(2.4) is unconditionally stable.[17]

Also, the convergence of the approximate finite difference scheme (2.1) -(2.4). Let $C(x_i, t_k)$ be the exact solution of the TFRDE (1.1)-(1.3) and C_i^k be the exact solution of the discrete equation (2.1)-(2.4) at the mesh point(x_i, t_k), *where* $i = 0,1,\dots,M-1; k = 1,2,\dots,N$.We define, $e_i^k = C(x_i, t_k) - C_i^k$, *where* $i = 0,1,\dots,M-1; k = 1,2,\dots,N$ and $E^k = (e_1^k, e_2^k, \dots, e_M^k)$

Now, we have, $E^0 = 0, E_0^k = 0$ *and* $E_N^k = 0$.

From (2.1), we get,

$$-re_{i-1}^1 + (1+2r)e_i^1 - re_{i+1}^1 = (1-\mu)e_i^0; \text{for } k = 0 \quad (4.1)$$

From (2.2), we get,

$$-re_{i-1}^{k+1} + (1+2r)e_i^{k+1} - re_{i+1}^{k+1} = (1-\mu-b_1)e_i^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-j} + b_k e_i^0;$$

for $k \geq 1$ (4.2)

where $r = \dfrac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}$; $\mu = \lambda\Gamma(2-\alpha)\tau^\alpha; b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$; $j = 0,1,2,\dots,k$;
$i = 0,1,2,\dots,M$ *and* $k = 0,1,2,\dots,N$.

Theorem 3.1.2 [17] The fractional order implicit finite difference scheme (2.1)-(2.4) for TFRDE (1.1)-(1.3) is convergent and the solution C_i^k of the discretize scheme (2.1)-(2.4) and the solution $C(x_i, t_k)$ of the equation (1.1)-(1.3) satisfy,

$$||C(x_i, t_k) - C_i^k|| \leq k||E||_\infty + O(\tau^{1-\alpha}, h^2); i = 0,1,\dots,M-1; k = 0,1,\dots,N$$

4. STUDY OF NUMERICAL SOLUTION OF THE SCHEME:

In this section, we obtain the approximated solution of TFRDE of both the Implicit and C-N finite difference schemes with initial and boundary conditions and compare their solutions. To obtain the numerical solution, it is important to use some analytical model. Thus, we present an example to demonstrate that TFRDE can be applied to simulate behavior of a fractional diffusion equation with the following parameters by using Mathematica Software. . We consider the following, dimensionless time fractional radon diffusion equation with suitable initial and boundary conditions.

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = D\frac{\partial^2 C(x,t)}{\partial t^2} - \lambda C(x,t), 0 < x < L; 0 < \alpha < 1; t > 0$$

initial condition: $C(x,0) = 0, 0 < x < L$

boundary conditions: $C(0,t) = c_0$ *and* $\frac{\partial C(x,t)}{\partial t} = 0, t \geq 0$

with the radon diffusion coefficient $D = 1.43 \times 10^{-6}\, Bq/m^3$. The numerical solutions obtained at t = 0.05 by considering the parameters $L = 1.7278cm$,

$$\lambda = 2.1 \times 10^{-6}s^{-1}, \tau = 0.05, k = 4\, m^2/kg, \rho = 0.5\, g/cm^3, C_0 = 200\, Bq/m^3,$$
$$C(0,t) = 40 \times 10^3, \quad \alpha = 0.9.$$

The rate of convergence of solutions of RDE of Implicit finite difference scheme respectively for $\alpha = 0.9\ nd\ 0.8$ is shown in table and graphically as in figure.

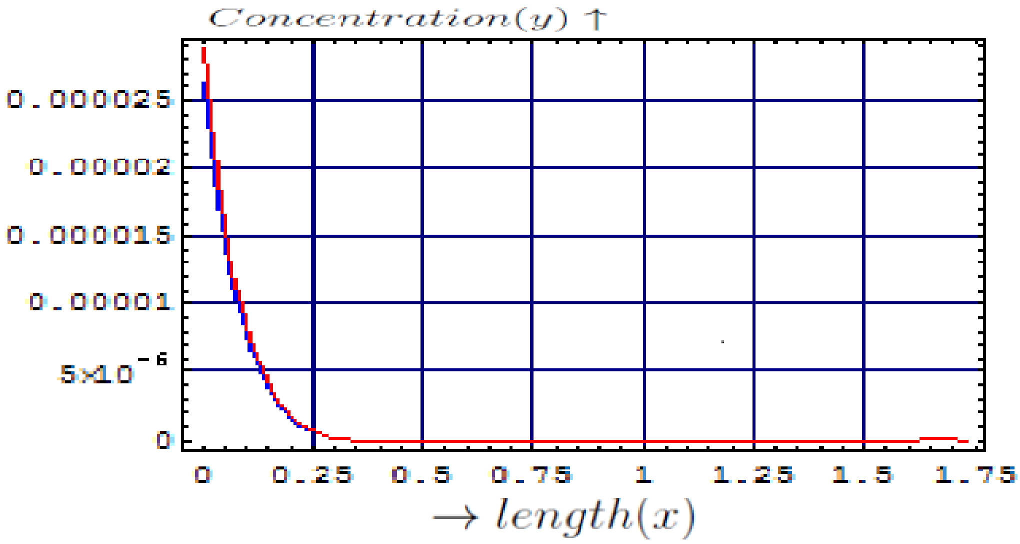

Figure . Rate of convergence of solution of radon diffusion equation for $\alpha = 0.9, 0.8$

5. CONCLUSION

We discuss the fractional order Implicit finite difference scheme for TFRDE and also, the stability and convergence of the scheme. As an application of this method we obtain the numerical solution of text problem and its solution is simulated graphically by mathematical software Mathematica.

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