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Best Proximity Pairs of Relatively Diametral Contractive Mappings

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Abstract : In this article, we present the idea of non-cyclic relatively diametral contractive mappings and prove that if $(\boldsymbol{A}, \boldsymbol{B})$ is a sharp proximal pair in a Banach space $\boldsymbol{X}$ with proximal normal structure and $\boldsymbol{T}: \boldsymbol{A} \cup \boldsymbol{B} \to \boldsymbol{A} \cup \boldsymbol{B}$ is a non-cyclic relatively diametral contractive mapping, then $\boldsymbol{T}$ has a best proximity pair in $\boldsymbol{A} \times \boldsymbol{B}$.

Keywords: Proximal normal structure, nonexpansive map, relatively diametral contractive mapping

1. INTRODUCTION

The geometric property normal structure was first introduced by Brodskii and Milman [2] in 1948. In the same year Kirk [1,8] proved that every nonexpansive self mapping on a weakly compact convex set K in a Banach space with normal structure has a fixed point in K.

Definition 1.1. Let K be a nonempty closed bounded convex subset of a Banach space X. A mapping $T: K \to K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in K$.

The concept of proximal normal structure was introduced by Eldred *et. al.* [4] in 2005, to study the existence of best proximity points for relatively nonexpansive mappings.

Definition 1.2. Suppose A and B is nonempty closed bounded convex sets in a Banach space X. A map $T: A \cup B \to A \cup B$ is called relatively nonexpansive map if $\|Tx - Ty\| \leq \|x - y\|, \forall x \in A, y \in B$.

So, if $A = B$, then T is nonexpansive on A. We say that a mapping $T: A \cup B \to A \cup B$ is non-cyclic if $T(A) \subset A, T(B) \subset B$.

Definition 1.3. Let A and B be closed bounded convex sets in a Banach space X and $T: A \cup B \to A \cup B$ be a non-cyclic mapping. A point $(x, y) \in A \times B$ is called a best proximity pair for T if $Tx = x, Ty = y$ and $\|x - y\| = \text{dist}(A, B)$.
Eldred *et. al.* [4] proved the existence theorem of best proximity pairs for a non-cyclic relatively nonexpansive mapping for weakly compact convex sets A and B in a strictly convex Banach space with having proximal normal structure. If a Banach space has proximal normal structure, then it has normal structure but the converse is not known [3,4]. Every uniformly convex Banach space has proximal normal structure [4]. Also if both A, B are compact convex sets in a Banach space, then the pair (A, B) has proximal normal structure [4]. In 2017, Dutta and Veeramani [3] showed that if A or B is compact in a Banach space, then the convex pair (A, B) has proximal normal structure. In that paper [3], they gave a characterization for proximal normal structure and also provided interesting examples of some Banach spaces with proximal normal structure. In subsequent years, many authors have studied proximal normal structure and relatively nonexpansive maps in their work (one may refer [3,5,6,9-11,13]).

In 2002, Taskovic [12] introduced a map called diametral contractive map and characterized normal structure for a weakly compact convex set of a Banach space in terms of fixed points of diametral contractive mappings.

Definition 1.4. Let K be a nonempty closed bounded convex set in a Banach space X. A mapping $T: K \to K$ is said to be diametral contractive if $\|Tx - Ty\| \leq \sup\{\|x-y\|: y \in C\}, \forall x, y \in C$, for every nonempty closed bounded convex subset C of K with $T(C) \subset C$.

Obviously every nonexpansive map on K is a diametral contractive map on K. Taskovic [12] established the next theorem for diametral contractive mappings.

Theorem 1.5. [12] Suppose K is a weakly compact convex set in a Banach space X and $T: K \to K$ is a diametral contractive mapping on K. Then K has normal structure if and only if T has a fixed point in K.

In this paper, we introduce the concept of non-cyclic relatively diametral contractive mapping as a generalization of non-cyclic relatively nonexpansive mappings and prove that if a sharp proximal pair (A, B) in a Banach space X with proximal normal structure and $T: A \cup B \to A \cup B$ is a non-cyclic relatively diametral contractive mapping on $A \cup B$, then T has a best proximity pair in $A \times B$.

2.Preliminaries

We state that a pair (A, B) in a Banach space X has a property if and only if both A and B have the same property. For illustration, (A, B) is closed if and only if A and B are closed. We also use the following notation:

$$(K_1, K_2) \subset (A, B) \Leftrightarrow K_1 \subset A, K_2 \subset B,$$
$$\delta(K_1, K_2) = \sup\{\|x-y\|: x \in K_1, y \in K_2\},$$
$$\text{dist}(K_1, K_2) = \inf\{\|x-y\|: x \in K_1, y \in K_2\},$$
$$\delta(\{x_n\}, \{y_n\}) = \delta(\{x_n: n \in \mathbb{N}\}, \{y_n: n \in \mathbb{N}\}).$$

Definition 2.1. Let (A, B) be a pair of sets in a Banach space X. The pair (A, B) is called proximal pair if for all $x \in A, y \in B$ there exist $x' \in A, y' \in B$ such that $\|x - y'\| = \|y - x'\| = \text{dist}(A, B)$.

Definition 2.2. [5] A pair of subsets (A, B) in a Banach space X is called a sharp proximal pair if for every $x \in A, y \in B$ there exist unique $x' \in A, y' \in B$ such that

$$\|x - y'\| = \|y - x'\| = \text{dist}(A, B).$$

Espilona [5] showed that in a strictly convex Banach space every proximal pair is a sharp proximal pair.

Definition 2.3. [2,7] Let C be a nonempty convex subset in a Banach space X. The set C is said to have normal structure if for each closed bounded convex subset K of C with $\text{diam}(K) > 0$, there exist $x \in K$ such that $r_x(K) < diam(K)$.

Definition 2.4. [4] Let(A, B) be a nonempty convex pair in a Banach space X. The pair (A, B) has proximal normal structure if for each closed bounded convex proximal pair $(C, D) \subset (A, B)$ for which $\text{dist}(C, D) = \text{dist}(A, B)$ and $\delta(C, D) > \text{dist}(C, D)$, there exist $x \in C, y \in D$ such that

$$r_x(D) < \delta(C, D), r_y(C) < \delta(C, D).$$

If each convex pair (A, B) in a Banach space X has proximal normal structure, then we say that the Banach space X has proximal normal structure. A convex pair (C, C) has proximal normal structure if and only if the convex set C has normal structure. Hence if a Banach space X has proximal normal structure, then it is also have normal structure.

The pursuing theorem gives a characterization for proximal normal structure.

Theorem 2.5. [3] Let (A, B) be a weakly compact convex pair in a Banach space X. The pair (A, B) does not have proximal normal structure if and only if there exist sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that $\|x_n - y_n\| = \text{dist}(A, B), \forall n, \|x_m - y_n\| > \text{dist}(A, B)$ for some m, n and

$$\lim_{n\to\infty} \text{dist}\,(y_{n+1}, co\{x_1, x_2, \dots, x_n\}) = \delta(\{x_n\},\{y_n\})$$

or,

$$\lim_{n\to\infty} \text{dist}\,(x_{n+1}, co\{y_1, y_2, \dots, y_n\}) = \delta(\{x_n\},\{y_n\}).$$

3. Main Results

In 2005, Eldred *et.al.*[4] proved the following result for the existence best proximity pairs for a non-cyclic relatively nonexpansive map.

Theorem 3.1. [4] Let (A, B) be a weakly compact convex pair of a strictly convex Banach space X. Suppose (A, B)has proximal normal structure and $T: A \cup B \to A \cup B$ is a non-cyclic relatively nonexpansive mapping. Then Thas a best proximity pair in $A \times B$.

In general a non-cyclic relatively nonexpansive map not necessarily be continuous on $A \cup B$, but in the case of Hilbert spaces the map is nonexpansive on $A \cup B$ and hence continuous on $A \cup B$ [4].

Definition 3.2. *Let* (A, B) *be a nonempty closed bounded convex pair of a Banach space* X. *A mapping* $T: A \cup B \to A \cup B$ *is said to be non-cyclic relatively diametral contractive if* $T(A) \subset A, T(B) \subset B$ *and for every nonempty closed bounded convex pair* $(C, D) \subset (A, B)$ *with* $T(C) \subset C, T(D) \subset D$, $\text{dist}(C, D) = \text{dist}(A, B)$, *we have*
$\|Tx - Ty\| = \text{dist}(A, B), \forall x \in C, y \in D$ *when* $\|x - y\| = \text{dist}(A, B)$,
$\|Tx - Ty\| \le r_x(D), \|Tx - Ty\| \le r_y(C), \forall x \in C, y \in D$ *when* $\|x - y\| > \text{dist}(A, B)$.

A non-cyclic relatively diametral contractive mapping on $A \cup B$is a non-cyclic relatively nonexpansive mapping on $A \cup B$. Although the converse is not true in general as shown by the next example.

Example 3.3. Consider $\mathbb{R}^2$ with the norm $\|.\|_2$.
Let $A = \{(0, t): t \in [0,1]\}$ and $B = \{(1, t): 0 \le t \le 1\}$.
Then(A, B) is a closed bounded convex pair in $(\mathbb{R}^2, \|.\|_2)$ with $\text{dist}(A, B) = 1$.

Define$T: A \cup B \to A \cup B$ as $T\big((0,t)\big) = \begin{cases} \left(0, \frac{t}{2}\right), t \ge \frac{1}{2} \\ \left(0, \frac{t}{4}\right), t < \frac{1}{2} \end{cases}$ for all$(0, t) \in A$ and

$T\big((1,t)\big) = \begin{cases} \left(1, \frac{t}{2}\right), t \ge \frac{1}{2} \\ \left(1, \frac{t}{4}\right), t < \frac{1}{2} \end{cases}$, for all$(1, t) \in B$.

Let $(C, D) \subset (A, B)$ be a nonempty closed bounded convex pair with $T(C) \subset C, T(D) \subset D$ and $\text{dist}(C, D) = \text{dist}(A, B)$.
If $x \in C, y \in D$ and $\|x - y\|_2 = \text{dist}(A, B)$, then there exists $t_0 \in [0,1]$ such that $x = (0, t_0), y = (1, t_0)$. Hence $\|Tx - Ty\|_2 = \text{dist}(A, B)$.
Now let $x \in C, y \in D$ such that $\|x - y\|_2 > \text{dist}(A, B)$, then there exist $t, s \in [0,1], t \ne s$ such that $x = (0, t), y = (1, s)$. Then $\|x - y\|_2^2 = 1 + |t - s|^2$.
Now if $s \ge \frac{1}{2}, t \ge \frac{1}{2}$ or $s \ge \frac{1}{2}, t \ge \frac{1}{2}$, then $\|Tx - Ty\|_2^2 \le \|x - y\|_2^2$

$$\Rightarrow \|Tx - Ty\|_2 \le r_x(D), \|Tx - Ty\|_2 \le r_y(C).$$

Let $t \ge \frac{1}{2}, s < \frac{1}{2}$. Then $\|Tx - Ty\|_2^2 = 1 + \left|\frac{t}{2} - \frac{s}{4}\right|^2 = 1 + \left(\frac{t}{2} - \frac{s}{4}\right)^2 \le 1 + t^2 \le r_x(D)$, since $(1,0) \in D$.
Now $\left(\frac{3}{4}\right) s \le \frac{t}{2} \Rightarrow \frac{t}{2} - \frac{s}{4} \le t - s$ and $\left(\frac{3}{4}\right) s > \frac{t}{2} \Rightarrow \frac{t}{2} - \frac{s}{4} < \left(\frac{3}{4}\right) s - \frac{s}{4} \le s$.
Therefore, $\|Tx - Ty\|_2^2 = 1 + \left|\frac{t}{2} - \frac{s}{4}\right|^2 = 1 + \left(\frac{t}{2} - \frac{s}{4}\right)^2 \le r_y(C)$, since $(0,0) \in C$.
The case is similar when $t < \frac{1}{2}, s \ge \frac{1}{2}$. Hence T is non-cyclic relatively diametral contractive on $A \cup B$.
Since $(\mathbb{R}^2, \|.\|_2)$ is a Hilbert space and T is not continuous on $A \cup B, T$ is not relatively nonexpansive on $A \cup B$.

Suppose (A, B) is a closed bounded convex pair in a Banach space X. Let $T: A \cup B \to A \cup B$ be a non-cyclic relatively diametral contractive mapping. A nonempty closed bounded convex pair $(K_1, K_2) \subset (A, B)$ with

$\text{dist}(A,B) = \text{dist}(K_1,K_2)$ and $T(K_1) \subset (K_1), T(K_2) \subset K_2$ is said to be a minimal invariant pair under T if for each nonempty closed bounded convex pair $(L_1,L_2) \subset (K_1,K_2)$ with $\text{dist}(L_1,L_2) = \text{dist}(K_1,K_2)$ and $T(L_1) \subset (L_1), T(L_2) \subset L_2$, we get $L_1 = K_1, L_2 = K_2$.

The next theorem states the existence of minimal invariant pairs for a non-cyclic relatively diametral contractive mapping.

Theorem 3.4. Suppose (A,B) is a weakly compact convex pair in a Banach space X and $T: A \cup B \to A \cup B$ is a non-cyclic relatively diametral contractive mapping. Then there exists a closed bounded convex proximal pair $(K_1,K_2) \subset (A,B)$ which is a minimal invariant pair under T.

Proof: Let $A_0 = \{x \in A: \|x-y\| = \text{dist(A,B)}, \text{ for some } y \in B\}$
$B_0 = \{y \in B: \|x-y\| = \text{dist(A,B)}, \text{ for some } x \in A\}$.
Since (A,B) is a weakly compact convex pair, the pair (A_0,B_0) is a nonempty weakly compact convex proximal pair with $\text{dist}(A_0,B_0) = \text{dist}(A,B)$.
Let $x \in A_0$. Then there exists a $y \in B$ such that $\|x-y\| = \text{dist}(A,B)$.
Therefore $\|Tx - Ty\| = \text{dist}(A,B)$. Hence $Tx \in A_0$. So, $T(A_0) \subset A_0$.
Similarly $T(B_0) \subset B_0$. Hence without loss of generality we assume that (A,B) is a proximal pair.
Let $\Gamma = \{(C,D) \subset (A,B): (C,D)$ is a nonempty closed bounded convex pair
with $\text{dist}(C,D) = \text{dist}(A,B)$ and $T(C) \subset C, T(D) \subset D\}$
Then Γ is nonempty, since $(A,B) \in \Gamma$. Define a partial order relation $'\leq'$ on Γ as $(C_1,D_1) \leq (C_2,D_2)$ if $C_2 \subset C_1, D_2 \subset D_1$. Let $\Gamma_\Lambda = \{(C_\alpha,D_\alpha): \alpha \in \Lambda\}$ be a chain in Γ. Let $C_0 = \cap_{\alpha\in\Lambda} C_\alpha$ and $D_0 = \cap_{\alpha\in\Lambda} D_\alpha$. Then by weak compactness, C_0 and D_0 are nonempty. Also,

$$T(C_0) \subset C_0, T(D_0) \subset D_0.$$

Now $(C_0,D_0) \subset (A,B)$ gives $\text{dist}(A,B) \leq \text{dist}(C_0,D_0)$. For each $\alpha \in \Lambda$, choose $x_\alpha \in C_\alpha, y_\alpha \in D_\alpha$ such that $\|x_\alpha - y_\alpha\| = \text{dist}(A,B)$. Now by weak compactness we can find weakly convergent subnets $\{x_{\alpha'}\}$ and $\{y_{\alpha'}\}$ such that $x_{\alpha'} \to^w x, y_{\alpha'} \to^w y$. Then $x \in C_0, y \in D_0$. Therefore,

$$\|x-y\| \leq \liminf_{\alpha'} \|x_{\alpha'} - y_{\alpha'}\| = \text{dist}(A,B).$$

So, $\text{dist}(A,B) = \text{dist}(C_0,D_0)$. Hence $(C_0,D_0) \in \Gamma$. Therefore, (C_0,D_0) is an upper bound of Γ. By Zorn's lemma there exists at least one $(K_1,K_2) \in \Gamma$ which is maximal relative to $'\leq'$, and hence it is a minimal invariant pair under T.
Now to show (K_1,K_2) is a proximal pair, define

$$K_1{}^0 = \{x \in K_1: \|x-y\| = \text{dist(A,B)}, \text{ for some } y \in K_2\}$$
$$K_2{}^0 = \{y \in K_2: \|x-y\| = \text{dist(A,B)}, \text{ for some } x \in K_1\}.$$

Then $(K_1{}^0, K_2{}^0) \subset (K_1,K_2)$ and $(K_1{}^0,K_2{}^0) \in \Gamma$. Hence by minimality of (K_1,K_2), we have $K_1{}^0 = K_1, K_2{}^0 = K_2$. Therefore (K_1,K_2) is a proximal pair.

The next theorem yields a property for minimal invariant pairs of sets.

Theorem 3.5. Suppose (A,B) is a weakly compact convex pair in a Banach space X. Let $T: A \cup B \to A \cup B$ be a non-cyclic relatively diametral contractive mapping. If (K_1,K_2) is a weakly compact convex minimal invariant pair under T, then $K_1 = \overline{co}\big(T(K_1)\big), K_2 = \overline{co}(T(K_2).)$

Proof: Let $L_1 = \overline{co}\big(T(K_1)\big), L_2 = \overline{co}\big(T(K_2)\big)$.
Now $T(K_1) \subset K_1, T(K_2) \subset K_2 \Rightarrow L_1 \subset K_1, L_2 \subset K_2 \Rightarrow T(L_1) \subset K_1 \subset L_1, T(L_2) \subset K_2 \subset L_2$.
Also, $\text{dist}(A,B) \leq \text{dist}(L_1,L_2) \leq \text{dist}\big(T(K_1),T(K_2)\big)$.
Since $\big(T(K_1),T(K_2)\big) \subset (K_1,K_2)$, $\text{dist}(K_1,K_2) \leq \text{dist}\big(T(K_1),T(K_2)\big)$. Let $x \in K_1, y \in K_2$ be such that $\|x-y\| = \text{dist}(K_1,K_2) = \text{dist}(A,B)$. Since T is a non-cyclic diameter contractive mapping, $\|Tx - Ty\| = \text{dist}(A,B)$. Hence $\text{dist}\big(T(K_1),T(K_2)\big) \leq \text{dist}(A,B) \Rightarrow \text{dist}\big(T(K_1),T(K_2)\big) = \text{dist}(A,B)$.
Hence $\text{dist}(A,B) = \text{dist}(L_1,L_2)$.
Therefore, by minimality of (K_1,K_2), $L_1 = K_1, L_2 = K_2$.

Theorem 3.6. Let (A, B) be a weakly compact convex sharp proximal pair in a Banach space X. Suppose $T: A \cup B \to A \cup B$ is a non-cyclic relatively diametral contractive mapping and(A, B) has proximal normal structure. Then T has a best proximity pair in $A \times B$.

Proof: Let (K_1, K_2) be a weakly compact convex minimal invariant pair under T.
Suppose $\delta(K_1, K_2) > \operatorname{dist}(K_1, K_2)$. Since (A, B) has proximal normal structure, there exist $x \in K_1, y \in K_2$ such that $r_x(K_2) < \delta(K_1, K_2)$ and $r_y(K_1) < \delta(K_1, K_2)$. Then there exists a real number r such that $r_x(K_2) \leq r < \delta(K_1, K_2)$ and $r_y(K_1) \leq r < \delta(K_1, K_2)$.
Since $(K_1, K_2)$is a proximal pair, there exist $x' \in K_2, y' \in K_1$ such that $\|x - x'\| = \operatorname{dist}(K_1, K_2), \|y' - y\| = \operatorname{dist}(K_1, K_2)$. Then $\frac{x+y'}{2} \in K_1, \frac{x'+y}{2} \in K_2$ and

$$\left\|\frac{x + y'}{2} - \frac{x' + y}{2}\right\| = \operatorname{dist}(K_1, K_2).$$

Define,

$$L_1 = \left\{p \in K_1 : r_p(K_2) \leq \frac{r + \delta(K_1, K_2)}{2}\right\}$$

$$L_2 = \left\{q \in K_2 : r_q(K_1) \leq \frac{r + \delta(K_1, K_2)}{2}\right\}.$$

Now $r_{\frac{x+y'}{2}}(K_2) \leq \frac{r_x(K_2)+r_{y'}(K_2)}{2} \leq \frac{r+\delta(K_1,K_2)}{2}$ and $r_{\frac{x'+y}{2}}(K_1) \leq \frac{r_{x'}(K_1)+r_y(K_1)}{2} \leq \frac{r+\delta(K_1,K_2)}{2}$.

Hence $\frac{x+y'}{2} \in L_1, \frac{x'+y}{2} \in L_2$. Therefore, (L_1, L_2) is a nonempty closed bounded convex pair with

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}(L_1, L_2).$$

Let $p \in L_1$. Then $r_{Tp}(K_2) = \sup\{\|Tp - q\| : q \in K_2\} = \sup\{\|Tp - q\| : q \in \overline{co}(T(K_2))\}$

$$= \sup\{\|Tp - q\| : q \in T(K_2)\} = \sup\{\|Tp - Tp'\| : p' \in K_2\}$$

$$\leq r_p(K_2) \leq \frac{r + \delta(K_1, K_2)}{2}.$$

Hence $T(L_1) \subset (L_1)$. Similarly, $T(L_2) \subset L_2$. Since $(K_1, K_2)$is a minimal pair, $L_1 = K_1, L_2 = K_2$.
Now, $\delta(K_1, K_2) = \delta(L_1, L_2) = \sup\{r_x(L_2) : x \in L_1\} \leq \frac{r+\delta(K_1,K_2)}{2} < \delta(K_1, K_2)$, which is a contradiction.
Therefore, $\delta(K_1, K_2) = \operatorname{dist}(K_1, K_2)$. Since $(K_1, K_2)$is a sharp proximal pair, $K_1 = \{x\}$ and $K_2 = \{y\}$. Hence $Tx = x, Ty = y$ and $\|x - y\| = \operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$.

Corollary 3.7. [12] Let K be a weakly compact convex subset of a Banach space X and $T: K \to K$ be a diametral contractive mapping on K and K has normal structure. Then T has a fixed point in K.
Proof: The proof follows from the Theorem 3.6, by taking $A = B = K$.

In Example 3.3., note that $(A, B)$is a sharp proximal pair with proximal normal structure and hence by Theorem 3.6, the map T has a best proximity pair in $A \times B$.

Example 3.7. Consider c_0 with the norm $\|x\| = |x(1)| + \sup\{|x(n)| : n \geq 2\}$.
Let $A = \{(0, t, 0,0, \ldots) : 0 \leq t \leq 1\}$ and $B = \{(1, s, 0,0, \ldots) : 0 \leq s \leq 1\}$.
Since (A, B) is a compact convex pair in $(c_0, \|.\|)$, (A, B) has proximal normal structure. Also, (A, B) is a sharp proximal pair with $\operatorname{dist}(A, B) = 1$ and $\delta(A, B) > 1$.
Then by Theorem 3.6., every non-cyclic relatively diametral contractive mapping on $A \cup B$ has a best proximity pair in $A \times B$.

4. Conclusion

In Theorem 3.6., we assume that the pair (A, B) is a sharp proximal pair and has proximal normal structure. We do not know that the proximal normal structure assumption in that Theorem can be dropped or not. However, the sharp proximal pair assumption can not be omitted. To see this:
Consider C_0 with $\|.\|_\infty$. Let $A = \{(0, t, 0,0, \ldots) : 0 \leq t \leq 1\}$ and $B = \{(1,0, s, 0, \ldots) : 0 \leq s \leq 1\}$.Then $\|x - y\|_\infty = \operatorname{dist}(A, B) = 1, \forall x \in A, y \in B$. Since (A, B) is a compact convex pair, (A, B) has proximal normal structure.

Let $p, q \in A$ such that $p \neq q$. Define $T: A \cup B \to A \cup B$ as $T(x) = \begin{cases} p, x \neq p \\ q, x = p \end{cases}, \forall x \in A$ and $T(x) = x, \forall x \in B$.Then $\|Tx - Ty\|_\infty = \|x - y\|_\infty = \mathrm{dist}(A, B), \forall x \in A, y \in B$. Hence T is a relatively diametral contractive mapping on $A \cup B$ without having a best proximity pair in $A \times B$. Note that(A, B) is not a sharp proximal pair.
So the following question arises.

Question: Is it possible to characterize proximal normal structure for a weakly compact convex sharp proximal pair (A, B) in a Banach space in terms of best proximity pairs of relatively diametral contractive mappings on $A \cup B$?

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