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APPROXIMATED SOLUTION OF FRACTIONAL ORDER MATHEMATICAL MODEL FOR TUBER CULOSIS WITH TWO LINE TREATMENT

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Abstract: In this work, we have proposed fractional order mathematical model constructed by six coupled fractional order ordinary differential equations of multi-drug resistant tuberculosis with twin line treatment and initial conditions. The present fractional order mathematical model has been solved by using fractional order Homotopy perturbation method which yield approximate solutions for all the six ordinary differential equations by applying Riemann-Leivoulli fractional integral. The properties and nature of physical states of these equations have been emphasized more precisely by taking fractional order. Numerical simulation of the model has been carried out by varying order of the differential equations under same value of parameters and uniform initial conditions. Convergence of series of approximate solution is prominently demonstrated. The graphical results have been obtained significantly which determines the impact of fractional order of the model. Fractional order of all ordinary differential equations in the mathematical model are considered to be uniform. The mathematical model with generalization of order has given new fundamental characteristics, deeper understanding and variety of results of tuberculosis. Fractional order model reduces the complexity and makes the complex structure easy for analysis of such natural phenomenons.

I. INTRODUCTION

In seventeenth century, researchers and scientists formulated various mathematical concepts in fractional order form. The concept of calculus on consideration of fractional order derivatives and integrals emerged as fractional calculus [1]-[2]. In the past decades, the systems of fractional order derivatives and integrations have been used in many fields of science and technology such as physics and astrophysics [3], Biomedical and biotechnology [4], mathematical modelling in biology [5] etc. Mathematical modelling for various diseases which governs diverse factors at various stages has brought wide interest among scientists on account of its significant role to provide excellent results and its dynamical behavior [6]-[8].

The inception of existence of Mycobacterium tuberculosis is doubtful, but some information about tuberculosis has been mentioned in ancient history of Egypt [9]. Danial [10] has given a brief history of tuberculosis and mentioned that the genus Mycobacterium tuberculosis originated more than 150 million years ago. Various factors like malnourishment, HIV infection, smoking, diabetics and use of alcohol are leading to new cases of tuberculosis. As per the recommendations of a recent modelling study, tuberculosis incidents could be gradually reduced by availing more social protection, creating awareness about preventive measures and eradicating extreme poverty. United Nations has highlighted the need for accelerating efforts to achieve the goal of ending tuberculosis epidemic by 2030 [11]-[12].

Mathematical modelling of tuberculosis with optimal control has been successfully represented and analysed by Fatmawati et. al. [13]. Muhafzn et. al. [14] have established a mathematical model of tuberculosis studied by. A fractional order mathematical model of tuberculosis with vaccination and reinfection have studied by Nandi et al. [15]. Arqub O. A. et. al. [16] have proposed fractional epidemic model and solved it by homotopy analysis method. In this paper, we have proposed fractional order homotopy perturbation method (FHPM). The method has successfully tackled the present fractional model and given rise to fruitful results. This method is the extension of homotopy perturbation method (HPM) proposed by He [17] and successfully implemented to solve various types of linear and non-linear differential equations. This method is a systematic combination of concept of topology called homotopy and classic perturbation technique [18]. This method provides suitable way to obtain approximate or analytic series solutions of wide range of problems in various fields.

II. PRELIMINARY

2.1. Basic definitions and some properties of fractional calculus.

In this segment, let's observe some basic definitions of fractional derivatives and integrals [18]-[19]

Theorem 2.1. [18] A real function $h(t), t > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exist a real number $p > \mu$ such that $h(t) = t^p\, h_1(t)$, where $h_1(t) \in [0\ \infty]$ and it is said to be in the space C_μ^n if and only if $h^n \in C_\mu, n \in N$.

Definition 2.2. (Riemann-Liouville fractional integral of order α)[18] Riemann-Liouville fractional integral operator $(({J_t}^\alpha))$ of order $\alpha \geq 0$ of a function $h \in C_\mu\ \mu \geq -1$ is defined as

$$ {}_a{J_t}^\alpha = \frac{1}{\Gamma\alpha}\int_a^t (t-\tau)^{\alpha-1}\, h(\tau)d\tau $$

$$ {}_a{J_t}^0 = h(t) $$

Where, $t \geq a \geq 0$ and Γ is Gamma function.

Some properties of Riemann-Liouville fractional integral operator are

For $h(t)\epsilon C_\mu,\ \mu\epsilon\ R\ , \mu \geq -1$ and $a, \alpha, \beta \geq 0, \nu \geq -1$

1.

$$ {}_a{J_t}^\alpha h(t).\, {}_a{J_t}^\beta h(t) = {}_a{J_t}^{\alpha+\beta} h(t) $$

2.

$$ {}_a{J_t}^\alpha h(t).\, {}_a{J_t}^\beta h(t) = {}_a{J_t}^\beta h(t).\, {}_a{J_t}^\alpha h(t) $$

3.

$$ {}_a{J_t}^\alpha (t-a)^\nu = \frac{\Gamma\,(\nu+1)}{\Gamma\,(\,\alpha+\nu+1)}(t-a)^{\,\alpha+\nu} $$

Definition 2.3. (Riemann-Liouville fractional derivative of order α) [18]

If $h(t)\epsilon C[a,b]$ and $a < t < b$ then

$$ {}_0{D_t}^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{h(\tau)}{(t-\tau)^\alpha}\, d\tau $$

Where $\alpha\epsilon(0,1)$ is called Riemann-Liouville fractional derivative of order α.

Definition 2.4. (Caputo fractional derivative) [18]

Caputo fractional derivative ${}_a{D_t}^\alpha$ of $h(t)$ is defined as

$$ {}_a{D_t}^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-\tau)^{-\alpha}\frac{d}{dt}h(\tau)\, d\tau $$

For $t \geq a \geq 0$ and $h\epsilon C_{-1}^n$

2.2. Fractional order homotopy perturbation method [FHPM]

In this section, we have briefly explained fractional order homotopy perturbation method to solve system of 'n' number of time fractional ordinary differential equations with initial conditions. The general form of system of time fractional order partial differential equations can be considered as follows

$$ {D^{\alpha_j}}\, u_j(x_1, x_2, x_3, \dots, x_{n-1}, t\,) + N_j\big(x_1, x_2, x_3, \dots, x_{n-1}, t\,, u_1, u_2, \dots, u_n\big) = g_j(x_1, x_2, x_3, \dots, x_{n-1}, t\,) \quad (2.1) $$

Where $x_j\epsilon\mathbb{R}^+$ and $j = 1,2, \dots, n$

With initial condition

$$ u_{i,0}(x_1, x_2, x_3, \dots, x_{n-1}, t_0\,) = f_i(x_1, x_2, x_3, \dots, x_{n-1}),\ i = 1,2, \dots, n \quad (2.2) $$

All N_j are nonlinear operators and g_j are functions of x_j and t.

By taking 'p' as an embedded parameter, we construct homotopy for each of the differential equation as follows

$$ (1-P)\big({D^{\alpha_j}}\, u_j - u_{j,0}\big) + P\begin{pmatrix} {D^{\alpha_j}}\, u_j + N_j\big(x_1, x_2, x_3, \dots, x_{n-1}, t\,, u_1, u_2, \dots, u_n\big) \\ +g_j(x_1, x_2, x_3, \dots, x_{n-1}, t\,) \end{pmatrix} = 0 \quad (2.3) $$

$$ \therefore {D^{\alpha_j}}\, u_j = u_{j,0} - P\begin{pmatrix} u_{j,0} + N_j\big(x_1, x_2, x_3, \dots, x_{n-1}, t\,, u_1, u_2, \dots, u_n\big) \\ +g_j(x_1, x_2, x_3, \dots, x_{n-1}, t\,) \end{pmatrix} \quad (2.4) $$

Applying inverse operator, ${t_0}^{{J_t}^{\alpha_j}}$ to both sides of Eq. (2.4), we get

$$ u_j(x_1, x_2, x_3, \dots, x_{n-1}, t\,) = {t_0}^{{J_t}^{\alpha_j}} u_{j,0} = P\,{t_0}^{{J_t}^{\alpha_j}}\begin{bmatrix} u_{j,0} + \\ N_j\big(x_1, x_2, x_3, \dots, x_{n-1}, t\,, U_1, U_2, \dots, U_n\big) \\ -g_j(x_1, x_2, x_3, \dots, x_{n-1}, t\,) \end{bmatrix} $$

Where, $j = 1,2, \dots, n$ and $U_j(x_1, x_2, x_3, \dots, x_{n-1}, t_0) = u_j(x_1, x_2, x_3, \dots, x_{n-1}, t_0)$

By equating the coefficients of power of P in (2.3), we get the series for the system (2.1) as

$$ U_j(x_1, x_2, x_3, \dots, x_{n-1}, t) = U_{j,0} + PU_{j,1} + P^2U_{j,2} + \cdots = \textstyle\sum_{i=0}^{\infty} P^i\ U_{j,i} \quad (2.5) $$

Where all $U_{j,i}$ are functions of $x_1, x_2, x_3, \dots, x_{n-1}$ and t. By taking $P \longrightarrow 1$ in Eq. (2.5), we have obtained the approximate series solution for the system (2.1) as

$$ U_j(x_1, x_2, x_3, \dots, x_{n-1}, t) = U_{j,0} + U_{j,1} + U_{j,2} + \cdots = \textstyle\sum_{i=0}^{\infty} U_{j,i} \quad (2.6) $$

It is necessary to note that the major advantage of fractional order homotopy perturbation method [FHPM] is that it gives solution in the form of perturbation series which can freely give solution and it may be convergence in all sense which has been explained independently.

III. ANALYSIS OF MATHEMATICAL MODEL OF TUBERCULOSIS

Now we have extended standard fractional order mathematical model for transmission of tuberculosis with twin line treatment of tuberculosis in human host which has been introduced by V. Kumar Gupta et. al.[21] to integer order form. To understand the dynamics of the disease, we have analysed system of fractional order ordinary differential equations which has defined model of tuberculosis with drug resistance to two line treatment. We have considered the human population constant. Let's define the present model by keeping uniform population 'N'. The total population will be divided into six classes. First one is a susceptible individuals with recruitment rate 'a', an individual may get infected which is represented by class of infected individuals I(t). Let the rate at which an individual will shift from susceptible to exposed individual E(t) be 'c' and exposed individual E(t) to infected individual I(t) be 'f'. On infection, a person goes for first line treatment $R_1(t)$ and second line treatment $R_2(t)$ with resistance rate to treatment 'h' and 'k' respectively. After proper and adequate treatment, the individuals will be shifted to recovered class $R(t)$. Since there is no permanent immunity to tuberculosis, recovered individual can be again transferred to susceptible disease class at the rate'd'. Considering the rates of disease roused mortality from class I(t), $R_1(t)$, $R_2(t)$ as 'g', 'l' and 'n' respectively. While rate of natural death is considered to be 'b'. The resistance class $R_1(t)$ and $R_2(t)$ on convalescence move to recovered class $R(t)$. at the rate 'm' and 'p' respectively. The proposed fractional order mathematical model of multi-drug resistant tuberculosis with two line treatment consist of six coupled fractional order differential equations.

$$0^{D_t^\alpha} S(t) = aN - bS(t) - c\,S(t)I(t) + d\,R(t) \quad (3.1)$$

$$0^{D_t^\alpha} E(t) = c\,S(t)I(t) + d\,R(t) - (b+f)E(t) \quad (3.2)$$

$$0^{D_t^\alpha} I(t) = f\,E(t) + d\,R(t) - (b+g+h+k)I(t) \quad (3.3)$$

$$0^{D_t^\alpha} R_1(t) = hI(t) - (b+l+m)R_1(t) \quad (3.4)$$

$$0^{D_t^\alpha} R_2(t) = kI(t) - (b+n+p)R_2(t) \quad (3.5)$$

$$0^{D_t^\alpha} R(t) = mR_1(t) + pR_2(t) - (b+d)R(t) \quad (3.6)$$

Where α is the real number such that $0 < \alpha \leq 1$

3.1. Approximated series solution of mathematical model.

In this section, we will apply FHPM to solve fractional order mathematical model for drug resistant tuberculosis with two line treatment. Let's take initial guess for the functions S(t), E(t), I(t), $R_1(t)$, $R_2(t)$ and R(t) and will construct homotopy for the equations (3.1) to (3.6).

Let $S_0(t) = S_0$, $E_0(t) = E_0$, $I_0(t) = I_0$, $R_{1_0}(t) = R_{1_0}$, $R_{2_0}(t) = R_{2_0}$ and $R_0(t) = R_0$ be initial guesses.

$$H(S(t),p) = (1-p)\left(0^{D_t^\alpha} S(t) - 0^{D_t^\alpha} S_0(t)\right) + p\left(0^{D_t^\alpha} S(t) - aN + bS(t) + cS(t)I(t) - dR(t)\right) = 0 \quad (3.7)$$

$$H(E(t),p) = (1-p)\left(0^{D_t^\alpha} E(t) - 0^{D_t^\alpha} E_0(t)\right) + p\left(0^{D_t^\alpha} E(t) - cS(t)I(t) + (b+f)R(t)\right) = 0 \quad (3.8)$$

$$H(I(t),p) = (1-p)\left(0^{D_t^\alpha} I(t) - 0^{D_t^\alpha} I_0(t)\right) + p\left(0^{D_t^\alpha} I(t) - fE(t) + (b+g+f+k)I(t)\right) = 0 \quad (3.9)$$

$$H(R_1(t),p) = (1-p)\left(0^{D_t^\alpha} R_1(t) - 0^{D_t^\alpha} R_{1_0}(t)\right) + p\left(0^{D_t^\alpha} R_1(t) - hI(t) + (b+l+m)R_1(t)\right) = 0 \quad (3.10)$$

$$(R_2(t),p) = (1-p)\left(0^{D_t^\alpha} R_2(t) - 0^{D_t^\alpha} R_{2_0}(t)\right) + p\left(0^{D_t^\alpha} R_2(t) - kI(t) + (b+n+p)R_2(t)\right) = 0 \quad (3.11)$$

$$(R_2(t),p) = (1-p)\left(0^{D_t^\alpha} R(t) - 0^{D_t^\alpha} R_0(t)\right) + p\left(0^{D_t^\alpha} R(t) - mR_1(t) - pR_2(t) + (b+d)R(t)\right) = 0 \quad (3.12)$$

Where 'p' is homotopy parameter such that $p \in [0,1]$

Let's suppose that the series solution of the equations (3.1) to (3.6) are as given below

$$\begin{aligned} S(t) &= S_0(t) + pS_1(t) + p^2S_2(t) + p^3S_3(t) + \cdots &&= \sum_{i=0}^{\infty} p^i S_i(t) \\ E(t) &= E_0(t) + pE_1(t) + p^2E_2(t) + p^3E_3(t) + \cdots &&= \sum_{i=0}^{\infty} p^i E_i(t) \\ I(t) &= I_0(t) + pI_1(t) + p^2I_2(t) + p^3I_3(t) + \cdots &&= \sum_{i=0}^{\infty} p^i I_i(t) \\ R1(t) &= R1_0(t) + pR1_1(t) + p^2R1_2(t) + p^3R1_3(t) + \cdots &&= \sum_{i=0}^{\infty} p^i R1_i(t) \\ R2(t) &= R2_0(t) + pR2_1(t) + p^2R2_2(t) + p^3R2_3(t) + \cdots &&= \sum_{i=0}^{\infty} p^i R2_i(t) \\ R(t) &= R_0(t) + pR_1(t) + p^2R_2(t) + p^3R_3(t) + \cdots &&= \sum_{i=0}^{\infty} p^i R_i(t) \end{aligned} \quad (3.13)$$

Substituting the series solution of the functions in their respective homotopy and equating coefficients of powers of p.

Let's take coefficients of p^0 as $S_0(t) = S_0$, $E_0(t) = E_0$, $I_0(t) = I_0$, $R1_0(t) = R1_0$, $R2_0(t) = R2_0$, $R_0(t) = R_0$

Equating the coefficients of p

$$S_1(t) = [aN - bS_0 - cS_0I_0 + dR_0]\frac{t^\alpha}{\Gamma(\alpha+1)} = A_0\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$E_1(t) = [cS_0I_0 - (b+f)E_0]\frac{t^\alpha}{\Gamma(\alpha+1)} = B_0\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$I_1(t) = [fE_0 - (b+g+h+k)I_0]\frac{t^\alpha}{\Gamma(\alpha+1)} = C_0\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$R1_1(t) = [hI_0 - (b+l+m)R1_0]\frac{t^\alpha}{\Gamma(\alpha+1)} = D_0\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$R2_1(t) = [kI_0 - (b+n+p)R2_0]\frac{t^\alpha}{\Gamma(\alpha+1)} = F_0\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$R_1(t) = [mRI_0 + pR2_0 - (b+d)R_0]\frac{t^\alpha}{\Gamma(\alpha+1)} = G_0\frac{t^\alpha}{\Gamma(\alpha+1)}$$

Equating the coefficients of p^2

$$S_2(t) = aN\frac{t^\alpha}{\Gamma(\alpha+1)} + [dG_0 - bA_0 - c(A_0I_0 + S_0C_0)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = aN\frac{t^\alpha}{\Gamma(\alpha+1)} + A_1\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$E_2(t) = [c(A_0I_0 + S_0C_0) - (b+f)B_0]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = B_1\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$I_2(t) = [fB_0 - (b+g+h+k)C_0]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = C_1\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$R1_2(t) = [hC_0 - (b+l+m)D_0]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = D_1\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$R2_2(t) = [kC_0 - (b+n+p)F_0]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = F_1\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$R_2(t) = [mD_0 + pF_0 - (b+d)G_0]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = G_1\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Equating the coefficients of p^3

$$S_3(t) = aN\frac{t^\alpha}{\Gamma(\alpha+1)} - aN(b+cI_0)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left[bA_1 + cI_0A_1 - cS_0C_1 + \frac{cA_0C_0\Gamma(2\alpha+1)}{(\Gamma(2\alpha+1))^2}dG_1\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = aN\frac{t^\alpha}{\Gamma(\alpha+1)} - A_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - A_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$E_3(t) = acNI_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left[c(A_1I_0 + cS_0C_1) - (b+f)B_1 + \frac{A_0C_0\Gamma(2\alpha+1)}{(\Gamma(2\alpha+1))^2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = B_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + B_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$I_3(t) = [fB_1 - (b+g+h+k)C_0]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = C_2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$R1_3(t) = [hC_1 - (b+l+m)D_1]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = D_2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$R2_3(t) = [kC_1 - (b+n+p)F_1]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = F_2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$R_3(t) = [mD_1 + pF_1 - (b+d)G_1]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = G_2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

Equating the coefficients of p^4

$$S_4(t) = aN\frac{t^\alpha}{\Gamma(\alpha+1)} - aN(b+cI_0)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left[(b+cI_0)A_2 + \frac{caNC_0\Gamma(2\alpha+1)}{(\Gamma(2\alpha+1))^2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left[(b+cI_0)A_3 - cS_0C_2 - \frac{c(A_0C_1+A_1C_0)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + dG_2\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$= aN\frac{t^{\alpha}}{\Gamma(\alpha+1)} - A_4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + A_5\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + A_6\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$E_4(t) = acNI_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left[cI_0A_2 - \frac{caN\,\Gamma(2\alpha+1)}{(\Gamma(2\alpha+1))^2}C_0 + (b+f)B_2\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$- \left[cI_0A_3 - cS_0C_2 + \frac{c(A_1C_0+A_0C_1)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + (b+f)B_2\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$- \left[cI_0A_3 - S_0C_2 + \frac{cC_1A_2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + (b+f)B_3 + dG_2\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$= \mathrm{B}_4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - B_5\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - B_4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$I_4(t) = f\mathrm{B}_2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + [B_3 - (b+g+h+k)C_2]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} = C_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + C_3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$R1_4(t) = [hC_2 - (b+k+m)D_2]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} = D_3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$R2_4(t) = [kC_2 - (b+n+p)F_2]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} = F_3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$R_4(t) = [mD_2 + pF_2 - (b+d)G_2]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} = G_3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

Equating coefficients of p^5

$$S_5(t) = aN\frac{t^{\alpha}}{\Gamma(\alpha+1)} - [aN(b+cI_o]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left[(b-cI_0)A_4 + \frac{aNC_0\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$- \left[(b+cI_0)A_5 - \frac{(c(A_2C_0+aNC_1)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + cS_0C_3\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$- \left[(b+cI_0)A_6 + \frac{(cA_3C_0\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)} + \frac{cA_1C_1\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + \frac{S_0C_2\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)} + cS_0C_4 + dG_3\right]\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$
$$= aN\frac{t^{\alpha}}{\Gamma(\alpha+1)} - A_7\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - A_8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - A_9\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - A_{10}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$E_5(t) = acNI_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left[cI_0A_2 - \frac{caC_0NT\,\Gamma(2\alpha+1)}{(\Gamma(2\alpha+1))^2} + (b+f)B_4\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$- \left[cI_0A_5 - \frac{(aNC_1+A_2C_0)c\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + cS_0C_3 - (b+f)B_5\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$- \left[cI_0A_6 - \frac{c(A_3C_0+A_0C_2)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)} + \frac{cA_1C_1\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2} + cA_1C_1 + cS_0C_4\ (b+f)B_6 + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{(A_0C_2)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(3\alpha+1)}\right]\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$
$$= \mathrm{B}_7\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - B_8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - B_9\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + B_{10}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$I_5(t) = f\left[\mathrm{B}_4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - B_5\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - B_6\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\right]$$
$$- (b+g+h+k)\left[C_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - C_4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\right]$$
$$= \mathrm{C}_5\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \mathrm{C}_6\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \mathrm{C}_7\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$R1_5(t) = hC_3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + [hC_4 - (b+k+m)]\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} = D_4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + D_4\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$R2_5(t) = kC_3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + [hC_4 - (b+n+p)]\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} = F_4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + F_4\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$R_5(t) = [mD_3 + pF_3 - (b+d)G_3]\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} = G_4\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

We get the approximate series solution of the system by taking $p \to 1$ in the equations 3.13

$$S(t) = S_0(t) + S_1(t) + S_2(t) + S_3(t) + \cdots = \sum_{i=0}^{\infty} S_i(t)$$
$$E(t) = E_0(t) + E_1(t) + E_2(t) + E_3(t) + \cdots = \sum_{i=0}^{\infty} E_i(t)$$
$$I(t) = I_0(t) + I_1(t) + I_2(t) + I_3(t) + \cdots = \sum_{i=0}^{\infty} I_i(t)$$
$$R1(t) = R1_0(t) + R1_1(t) + R1_2(t) + R1_3(t) + \cdots = \sum_{i=0}^{\infty} R1_i(t)$$
$$R2(t) = R2_0(t) + R2_1(t) + R2_2(t) + R2_3(t) + \cdots = \sum_{i=0}^{\infty} R2_i(t)$$
$$R(t) = R_0(t) + R_1(t) + R_2(t) + R_3(t) + \cdots = \sum_{i=0}^{\infty} R_i(t)$$

3.2. **Convergence of fractional order homotopy perturbation method [FHPM].**

In this section, we need to prove the convergence of series of functions since the solution of functions are in the form of series [20].

Definition 3.1. (Convergence of Series)
Let $v_1(x), v_2(x), v_3(x) \ldots$ be a sequence of real valued functions. The series $\sum_{n=0}^{\infty} v_n(x)$ is said to converge to the function $v(x)$, if the sequence $\{v(x)\}_{n=0}^{\infty}$ of partial sums defined by
$N(x) = \sum_{n=1}^{\infty} v_n(x)$ converge to $v(x)$ [16] .
Theorem 3.2. Suppose X and Y be Banach spaces and $N: X \to Y$ is a contradiction nonlinear mapping, that is for all $v, \bar{v} \in X$ [20]
$\|N(v) - N(\bar{v}\| \leq \gamma \|v - \bar{v}\| \qquad 0 < \gamma < 1$
Which is according to Banach fixed point theorem having fixed point such that
$N(u) = u.$
The sequence generated by NHPM is regarded as
$V_{n-1} = N(V_{n-1}) \qquad V_{n-1} = \sum_{i=0}^{n-1} u_i \qquad$ n=1, 2, 3 ….
And suppose that $V_0 = v_0 \in B_r(u)$, where $B_r(u) = \{u^* \in X \| u^* - v\| \leq r\}$, then
We have the following statement
1) $\|V_n - u\| \leq \gamma^n \|v_0 - u\|$
2) $V_n \in B_r(u)$
3) $\lim_{n\to\infty} V_n = u$

In this section, we have proved convergence of the approximate series solution of all Classes. As by initial guesses,
$S_0(t) = S_0$, $E_0(t) = E_0$, $I_0(t) = I_0$, $R1_0(t) = R1_0$, , $R2_0(t) = R2_0$, $R_0(t) = R_0$ are constants, hence are bounded functions in $[0, t_0]$.
$|S_1(t) \leq A_0|$, $|E_1(t)| \leq B_0$, $|I_1(t)| \leq C_0$
$R1_1(t) \leq D_0$, $R2_1(t) \leq F_0$, $R_1(t) \leq G_0$
Consequently, $S_i(t)$, $E_i(t)$, $I_i(t)$, $R1_i(t)$, $R2_i(t)$, and $R_i(t)$ for i=2,3,4….. are also bounded in domain $[0, t_0]$. Hence by theorem 3.2, the series solutions for all the classes are converge such that

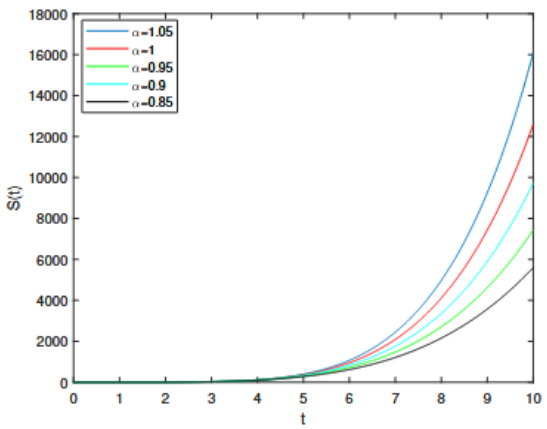
$$\sum_{i=0}^{\infty} S_i(t) = S(t), \quad \sum_{i=0}^{\infty} E_i(t) = E(t), \quad \sum_{i=0}^{\infty} I_i(t) = I(t)$$

$$\sum_{i=0}^{\infty} R1_i(t) = R1(t), \quad \sum_{i=0}^{\infty} R2_i(t) = R2(t), \quad \sum_{i=0}^{\infty} R_i(t) = R(t)$$

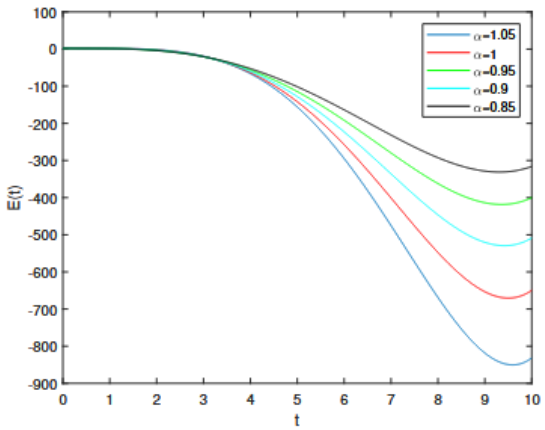
The above theory proves convergence of series solution of the mathematical model by FHPM.

Iv. NUMERICAL IMPLEMENTATION

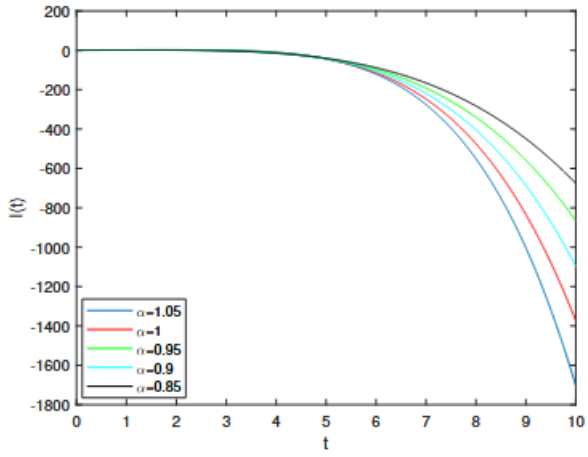
In this section, numerical simulation of approximate solution in the form of series of present model by fractional order homotopy perturbation method [FHPM] has been presented for all classes in the form of 2-D plots. To have a pragmatic approach based on practical rather than theoretical considerations two sets of initial values and parameters are fitted for explanation of nature of classes. The graphical results have been analysed for various values of order of differential equations.



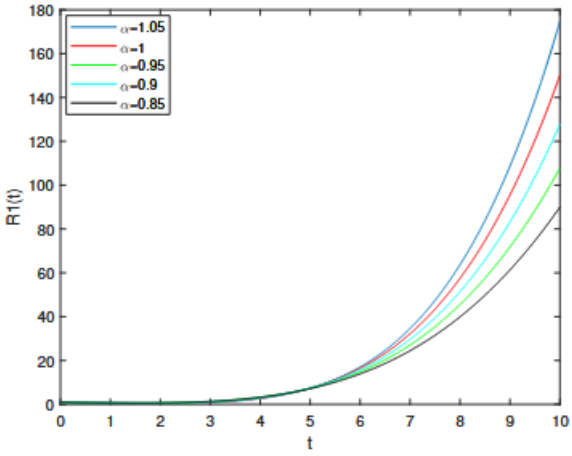
(a) Graph of S(t) against time t varying α



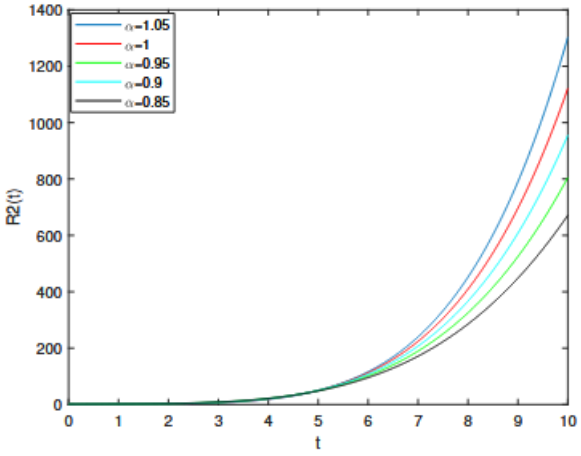
(b) Graph of E(t) against time t varying α



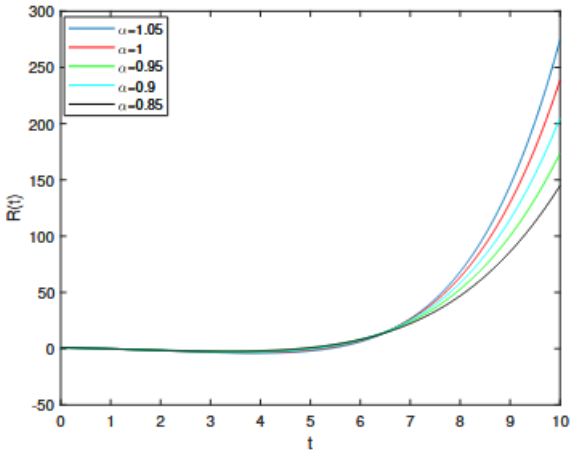
(c) Graph of I(t) against time t varying α



(d) Graph of R1(t) against time t varying α



(e) Graph of R2(t) against time t varying α



(f) Graph of R(t) against time t varying α

Figure 1. Graphs of all classes against time domain by varying the order (α) of derivatives.

4.1. Discussions. In the first case, the initial conditions are fitted as S(0) = 1, E(0) = 2, I(0) = 1, R1(0) = 1, R2(0) = 1, R(0) = 1, and the values of parameters used are a = 0.0007, N = 1000, b = 0.07, c = 0.7, d = 0.75, f = 0.998, g = 0.008, h = 0.07, k = 0.55, l = 0.0004, m = 0.008, p = 0.0095, n = 0.3, atα = 1.05, α = 1, α = 0.95, α = 0.9, α = 0.85.

Figure 1(a) describes susceptible individuals S(t) against time t for varying α. It is observed that for some period of time, there is steady position of S(t) followed by considerable increase in number of susceptible individuals. More gain has been observed with respect to increase in order of differential equation i.e. α.

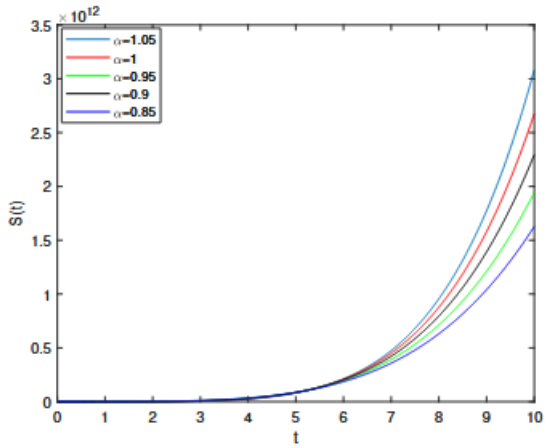
Figure 1(b) defines the nature of exposed individuals E(t) against time t for varying α. It communicates decline in exposed individuals after keeping constant state at initial condition. It has been observed that rate of exposed individuals is directly proportional to order of differential equation of E(t) i.e. α. The graph shows some positive inclination in the last stage, as some cases may be exposed after recovery too.

Figure 1(c) defines the behaviour of infected individuals I(t) against time t for varying α which emphasizes sharp decline in infected individuals I(t) and it is to be noted that the changes in the graphs positively map with the change in order of equation 3.3 of infected individuals.

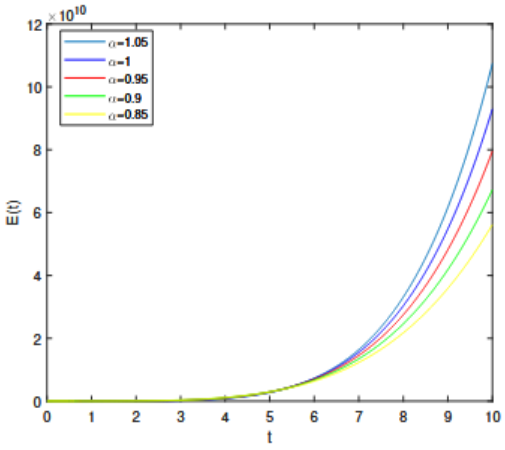
Figure 1(d) shows the graph resistance to first line treatment individuals R1(t) against time t for varying α takes stationery state up to certain state, then it increases slowly as compared to infected individuals. With reference to relation of order α of differential equation to R1(t) i.e. 3.4, increase in R1(t) takes place as the increase in α.

Figure 1(e) depicts gradual increase in resistance to second line treatment individuals R2(t) against time t for varying α. It is to be noted that the positive inclination is found more in R2(t) than R1(t).

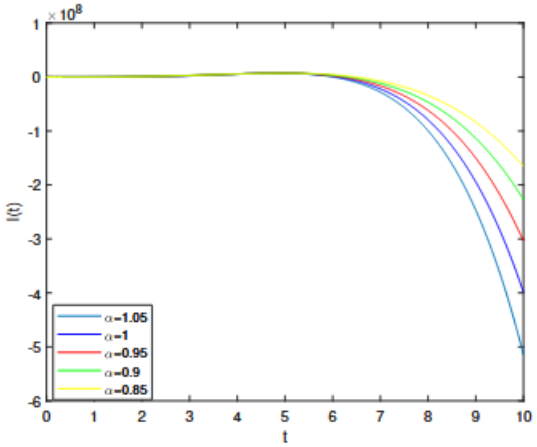
Figure 1(f) shows class of recovered individuals R(t) who have been treated eliciting positive trend as time increases. Rate of change of recovered individual takes steady position up to some period of time then it increases gradually as per the order α of differential equation 3.6 of recovered individual class .



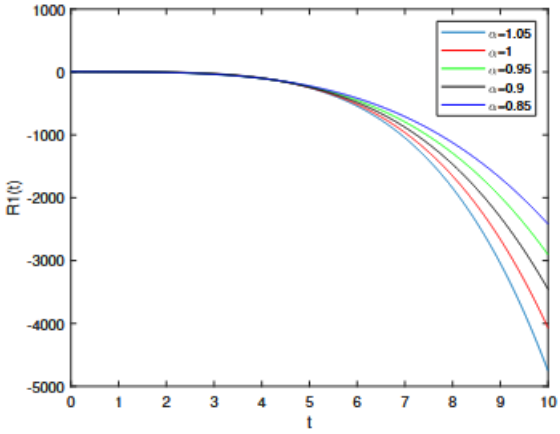
(a) Graph of S(t) against time t varying α



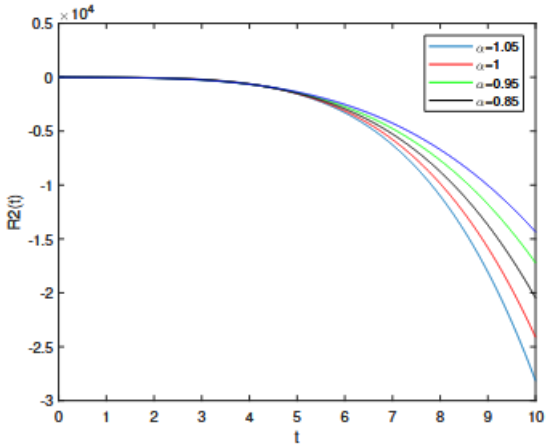
(b) Graph of E(t) against time t varying α



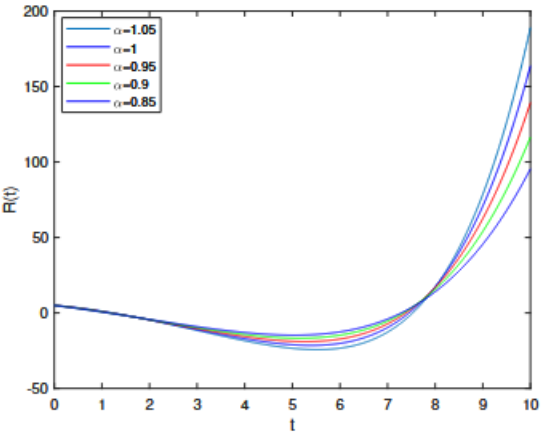
(c) Graph of I(t) against time t varying α



(d) Graph of R1(t) against time t varying α



(e) Graph of R2(t) against time t varying α



(f) Graph of R(t) against time t varying α

Figure 2. Graphs of all classes against time domain by varying the order (α) of derivatives.

In the next set, the total population is fitted as 10000 and initial conditions for number of individuals of all classes have been estimated and considered as S(0) = 2000 E(0) = 50, I(0) = 50, R1(0) = 5, R2(0) = 5, R(0) = 5. All the parameters represent the recruitment rate of individual from one type of class to another type of class. We have considered their respective values as a = 0.00005, b = 0.56, c = 0.15, d = 0.0075, f = 0.08, g = 0.008, h = 0.0009, k = 0.0055, l = 0.04, m = 0.00001, p = 0.0095, n = 0.03 to get graphical results for all classes in figure 2.

Figure 2(a) describes susceptible individuals S(t) against time (t) for α = 1.05, α = 1, α = 0.95, α = 0.9, α = 0.85. The plots defines the characteristics of susceptible individuals (S(t)) as for some period it keeps steady position, followed by considerable increase in number of susceptible individuals.

Figure 2(b) defines the inherent traits of exposed individuals E(t) against time t for α. It communicates inclination of graph succeeding steady state at initial stage. It has been observed that rate of exposed individuals is directly proportional to order of differential equation of E(t).

Figure 2(c) defines the trends in infected individuals I(t) against time (t) for α = 1.05, α =1, α = 0.95, α = 0.9, α = 0.85 which underline decrease in infected individuals I(t).

Figure 2(d) and 2(e) depicts the plots of individuals resistant to first line treatment individuals R1(t) and resistant to second line treatment individuals R2(t) against time t respectively. The plots are considered for α = 1.05, α = 1, α = 0.95, α = 0.9, α = 0.85. While the graphs are stationary upto certain time, decrement takes place progressively. This decrement is on account of positive effect of treatment leading to recovery.

Figure 2(f) represents class of recovered individuals R(t) who have been treated successfully. The plot has a variable pattern as there is steady state followed by a short phase of declination with respect to varying α. Therefore in the above figures 1 and 2, we strongly declare that the proposed fractional order mathematical model for tuberculosis with two line treatment with initial conditions and stated values of parameters evaluates the system more efficiently. The fractional order model focuses on effect of nonlocal frequency under laws of natural phenomenons in tuberculosis.

V. CONCLUSION

Simulation of approximate solution of fractional order mathematical model has been carried out for different values of order (α) of differential equations present in the model taking same value of parameters and initial guesses of individual classes. It has been observed that the most important role of the present model is to provide an excellent instrument for the description of dynamic behavior, memory status and hereditary properties to deal with its physical processes. The fractional order mathematical models explain the current status along with the historical background of the system. The mathematical model with generalization of order has given new fundamental characteristics, deeper understanding and variety of results of tuberculosis. Fractional order model reduces the complexity and makes the complex structure easy for analysis of such natural phenomenons. It is clear that simulation results give the similar characters as with the mathematical model of SIR with vital dynamics and constant population [20] but the present results are more dynamic.

VI. ACKNOWLEDGMENT

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