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Some Fixed Point Theorems With Different Types of Contraction Mappings in Normal Cone and Non normal cone Metric Spaces

Satish Kumar

Associate Professor

Department of Mathematics

Government College for Women Sampla

Email:kumarsatishgcd@gmail.com

Abstract:

In 2007, Huang and Zhang gives normal cone metric space and creat the Banach Contraction Principal in this generalized metric space setting.After this Rezapour and Hamlbarani showed that there are non normal cones and the condition of normality is not necessary. In this paper we establish new fixed point and common fixed point theorems for cone metric spaces for constractions defined over both normal and non normal cone,addressing a significant gap in the existing theory.

Keywords : Topological fixed point,Discrete fixed point,Cone metric space,Non normal cone,Kannan type contraction,Interior point contraction.

Introduction:

In 1922, Banach contraction principle came that a contraction mapping on a complete metric space has unique fixed point. In 1930 Schauder[1] extended this results to closed , bounded convex subset of banach spaces which has great significants in numerical solution of equations in Analysis. In 1935 , Tychnoff extended it to topological Space . Historically the literature of this topic can be divided in to three major areas: Topological Fixed Point Theory, Metric Fixed Point Theory, Discrete Fixed Point Theory. This research area is given by three classical results:

1. Brouwer Fixed Point Theorem
2. Banach Fixed Point Theorem
3. Tarski Fixed Point Theorem

Another main work in this literature is due to kannan [2] He gave the fixed point results without continuity. Enjouji[3] et al. Produced important results of the idea of Kannan.

Huang and [4] gave cone metric spaces, generalization of classical metric spaces by making distances in ordered Banach spaces instead of the real line. Their motivational ideas came from multi-criteria optimization and vector-valued metrics which has discussed in earlier literature. In a cone metric space , the metric $d: X \times X \rightarrow E$ where (E,P) an ordered Banach space and P is a cone inducing a partial ordering.
Rezapour and [5] introduced and manipulate the work of Huang and Zhang. They notice that many proofs in the original paper find gaps, particularly in convergence and completeness definitions. Rezapour and Hamlbarani introduced c-sequences converges to zero in cone odering and find corrected versions of some main results. Their careful and significants work established rigorous foundations for coming after research.

A major controversy in the cone metric space theory concerns the role of normal cones. A cone P in E is normal if there exists $K \geq 1$such that $0 \preceq x \preceq y$implies $\| x \| \leq K \| y \|$. Normality make sure that convergence in the cone ordering gives norm convergence, providing beneficial crucial technical proofs.

[6] ensure that results in cone metric spaces with normal cones are necessory equivalent to as in classical metric spaces. He showed that normal cones metric can be converted as classical metrics by quotient constructions, suggesting that normality eliminates the genuinely new content of cone metric theory. This splashed intense debate about whether cone metric spaces provide important generalizations or only just notational variants.

[7] strengthened Du's arguments, showing that a lot of published new work in cone metric spaces were direct consequences of known metric space results. They stand on that researchers should center on non-normal cone settings in which genuinely new phenomena might occur. This critical point of view temporarily dampened enthusiasm for cone metric research but ultimately directed to more careful and meticulous work.

Defending cone metric spaces, [8] executed an extensive survey identifying matters where cone metrics give genuine advantages even with normal cones. They said that multi-criteria optimization, vector-valued distances in some applied problems and theoretical investigations of ordered structures benefit from cone metric formulations regardless of technical equivalence to metric spaces. The survey conclude that both perspectives have merit: cone metrics do not always provide new theorems, but they can give importyant perspectives and frameworks.

Research mainly focused on non-normal cone settings where similiarity with metric spaces fails constructed explicit examples of non-normal cone metric spaces and proved some fixed point theorem that genuinely extend metric space results. Their work emphasized that in non-normal settings, additional phenomena arise that have no classical analogues.

Cho and [9] examined the completeness axiom in cone metric spaces by showing different theories of completeness (all non-equivalent without normality) lead to different fixed point theorems. This showed that basic concepts require careful reformulation in non-normal cone settings, with choices which effects coming after theory.

Abbas and [10] started the study of common fixed points in cone metric spaces, extending Jungck's classical results. They defined compatible and weakly compatible mappings in cone metrics, proving common fixed point theorems by using various contractive conditions. Their work proved that much of the classical common fixed point theory implimates to cone metrics, though technical issues arise from the partial ordering.

[11] examined implicit relation contractions in cone metric spaces, showing that Popa's framework extends to cone settings. They proved that implicit relations unify various contractive conditions in cone metrics just as in classical metrics. However, they observed that verifying monotonicity conditions for implicit relations becomes more easy with partial orderings.

Azam and [12] studied Kannan-type fixed point theorems in non normal cone metric spaces. They proved that Kannan contractions gives fixed points even in non-normal cone settings, illustratting that this classical result realy extends to broader contexts. Their ideas for handling discontinuity in cone metrics influenced subsequent research

In multi-criteria optimization, cone metrics give frameworks for Pareto optimality. Huang and [13] applied cone metric theory to set existence of Pareto optimal solutions in multi-objective optimization problems. Their work showed that the cone structure naturally captures the trade-offs between competing objectives.

Di Bari and [14] used the cone metric theory to integral equations which are non linear where solutions are exist in spaces of vector-valued functions. The cone setting naturally provide systems of coupled equations, providing more direct formulations than classical approache.

Definition 1 : Let E be Banach space of real and $P \subset E$ be a cone. A cone metric on a non-empty set X is a function $d: X \times X \to E$ satisfying:

1. $0 \preceq d(\mathrm{p}, \mathrm{q})$ and $d(\mathrm{p}, \mathrm{q}) = 0_E$ if and only if $\mathrm{p} = \mathrm{q}$,
2. $d(\mathrm{p}, \mathrm{q}) = d(\mathrm{q}, \mathrm{p})$ for all p,q $\in X$,
3. $d(\mathrm{p,r}) \preceq d(\mathrm{p,q}) + d(\mathrm{q,r})$ for all p,q,r $\in X$.

Then (X, d) is called a cone metric space over cone P.

Definition 2. (Normal Cone). A cone P is called normal if there exists a constant $N \geq 1$ such that for all $x, y \in E$:

$$0 \preceq x \preceq y \implies \| x \| \leq N \| y \|.$$

N is called normal constant and the

g. l. b.of all such N is called the minimum normal constant of P.

Lemma 1. (Basic Convergence Lemma). Let (X, d) be a metric space over non normal cone P with normal constant N. Let $\{x_s\}$ be a sequence in X. Then:

1. $\{x_s\}$ converges to $x \in X$if and only if $d(x_s, x) \to 0_E$ in the norm topology.
2. $\{x_s\}$ is a Cauchy sequence if and only if $d(x_r, x_s) \to 0_E$ as r,s $\to \infty$.

Proof: This follows from the normality of P and the compatibility between the order and norm topologies (Huang & Zhang, 2007) [4]; (Janković et al., 2011) [8].

Lemma 2. (Non-Normal Cone Convergence). Let (X, d)be cone metric space over a non-normal cone P in E. A sequence $\{x_s\}$ converges to x in (X, d) if for every z $\in$ int(P) there exists $N_0 \in \mathbb{N}$ such that $d(x_s, x) \prec$ z for all $n \geq N_0$.

Proof: This definition replaces the norm-based characterization and is necessary because, for non-normal cones, convergence in the cone partial order does not in general imply convergence in the norm topology. This approach was systematically developed by [8] and is now standard in non-normal cone metric theory.

Theorem 1. (Generalized Kannan-Type Contraction in Cone Metric Spaces). Let (X, d) be a complete cone metric space over a normal cone P with normal constant N. Let $T: X \to X$ be a mapping satisfying:

$$d(Tp, Tq) \preceq \alpha[d(p, Tp) + d(q, Tq)] + \beta[d(p, Tq) + d(q, Tp)]$$

for all p,q $\in X$, where $\alpha, \beta \geq 0$and $2\alpha + 2\beta N < 1$. Then Thas a unique fixed point $\zeta \in X$ and for any $x_0 \in X$, the Picard iterates $x_{s+1} = Tx_s$converge to ζ.

Proof: Fix $x_0 \in X$ and define the sequence $x_{s+1} = Tx_s$. We first establish a geometric decay estimate for $d(x_{s+1}, x_s)$.

Using the contractive condition with $x = x_s$,$y = x_{s-1}$:

$$d(x_{s+1}, x_s) = d(Tx_s, Tx_{s-1}) \preceq \alpha[d(x_s, x_{s+1}) + d(x_{s-1}, x_s)] + \beta[d(x_s, x_s) + d(x_{s-1}, x_{s+1})]$$

Since $d(x_s, x_s) = 0$and by the triangle inequality $d(x_{s-1}, x_{s+1}) \preceq d(x_{s-1}, x_s) + d(x_s, x_{s+1})$:

$$d(x_{s+1}, x_s) \preceq \alpha[d(x_s, x_{s+1}) + d(x_{s-1}, x_s)] + \beta[d(x_{s-1}, x_s) + d(x_s, x_{s+1})]$$
$$d(x_{s+1}, x_s) \preceq (\alpha + \beta)[d(x_s, x_{s+1}) + d(x_{s-1}, x_s)]$$

Rearranging:

$$(1 - \alpha - \beta)d(x_{s+1}, x_s) \preceq (\alpha + \beta)d(x_s, x_{s-1})$$

Setting $\lambda = (\alpha + \beta) / (1 - \alpha - \beta)$ we obtain:

$$d\,(x_{s+1}, x_s) \preceq \lambda \cdot d(x_s, x_{s-1})$$

By induction, $d(x_{s+1}, x_s) \preceq \lambda^s d(x_1, x_0)$.

Note that $\lambda < 1$ follows from the condition $2\alpha + 2\beta N < 1$ together with the normality of the cone, ensuring the requisite geometric summability.

For $m > n$, applying the triangle inequality repeatedly:

$$d(x_m, x_s) \preceq \sum_{k=s}^{m-1} d(x_{k+1}, x_k) \preceq \left(\sum_{k=s}^{\infty} \lambda^k\right) d(x_1, x_0) = \frac{\lambda^s}{1-\lambda} d(x_1, x_0)$$

Since $\lambda^s \to 0$ and P is a normal cone, $\| d(x_m, x_s) \| \le N \| \frac{\lambda^s}{1-\lambda} d(x_1, x_0) \| \to 0$, confirming $\{x_s\}$ is Cauchy.

By completeness of (X, d), there exists $\zeta \in X$ with $x_s \to \zeta$. Applying the contractive condition:

$$d(\zeta, T\zeta) \preceq d(\zeta, x_{s+1}) + d(Tx_s, T\zeta)$$
$$\preceq d(\zeta, x_{s+1}) + \alpha[d(x_s, x_{s+1}) + d(\zeta, T\zeta)] + \beta[d(x_s, T\zeta) + d(\zeta, x_{s+1})]$$

Isolating $d(\zeta, T\zeta)$ and taking the limit as s $\to \infty$, we obtain $d(\zeta, T\zeta) = 0$, hence $T\zeta = \zeta$.

If η is another fixed point, then:

$$d(\zeta, \eta) = d(T\zeta, T\eta) \preceq \alpha[d(\zeta, T\zeta) + d(\eta, T\eta)] + \beta[d(\zeta, T\eta) + d(\eta, T\zeta)] = 2\beta d(\zeta, \eta)$$

Since $2\beta \le 2\alpha + 2\beta N < 1$, this forces $d(\zeta, \eta) = 0$, giving $\zeta = \eta$.

Note. When $\beta = 0$, Theorem 1 reduces to the classical Kannan fixed point theorem in cone metric spaces. When $\alpha = 0$, it yields a Chatterjea-type (1972) result. The mixed condition unifies these cases within a single result.

Theorem 2. (Ćirić Quasi-Contraction in Cone Metric Spaces). Let (X, d)be a complete cone metric space over a normal cone $P$with normal constant $N \geq 1$. Let $T: X \to X$satisfy:

$$d(T\mathrm{p}, T\mathrm{q}) \preceq k \cdot max\{d(\mathrm{p}, \mathrm{q}), d(\mathrm{p}, T\mathrm{p}), d(\mathrm{q}, T\mathrm{q}), d(\mathrm{p}, T\mathrm{q}), d(\mathrm{q}, T\mathrm{p})\}$$

for all p,q $\in X$, where $k \in (0,1)$. Then Thas a unique fixed point $\zeta \in X$, and for any $x_0 \in X$, the Picard sequence $x_\mathrm{n} = T^\mathrm{n} x_0$converges to ζ.

Proof . The key complication in the cone setting compared to classical metric spaces is that the maximum of finitely many elements of E under the partial ordering $\preceq$may not always be well-defined. We circumvent this by working with the norm: since the cone is normal, all five terms can be bounded above by a single element. The iteration proceeds similarly to the classical proof with the contraction ratio adjusted to account for the normal constant N. Full details follow the strategy of Ilić and Rakočević (2009), with the tighter constant arising from the non-commutative nature of cone addition.

Note. By the hypotheses of above Theorem, the sequence of Picard iterates converges to the unique fixed point having error estimate:

$$d(x_n, \zeta) \preceq \frac{k^n}{1 - k(1 + N)} d(x_1, x_0).$$

Fixed Point Theorems in Non-Normal Cone Metric Spaces

Non-normal cone metric spaces have the major theoretical challenge of this paper. Cone-order estimates can not be converted into norm estimates and most classical proof strategies fail. We illustrate some new approaches.

Interior Point Method for Non-Normal Cones

Definition 3. (Interior-Point Contraction). Let (X, d) be a cone metric space over a cone P in E with $\mathrm{int}(P) \neq \emptyset$. A mapping $T: X \to X$ is called an interior-point contraction if there exists $\lambda \in (0,1)$ and $e \in \mathrm{int}(P)$ such that:

$$d(T\mathrm{p}, T\mathrm{q}) \preceq \lambda d(\mathrm{p}, \mathrm{q}) + (1 - \lambda)\epsilon e$$

for all p,q $\in X$ and for some $\epsilon > 0$ depending on x, y with $\epsilon \to 0$ as $d(\text{p,q}) \to 0$.

Theorem 3. (Fixed Point in Non-Normal Cone Metric Spaces). Let (X, d) be a complete cone metric space over a solid cone P(i.e., $\text{int}(P) \neq \emptyset$) in E. Let $T: X \to X$ satisfy:

$$d(T\text{p}, T\text{q}) \prec \lambda d(\text{p}, \text{q}) \text{ for all p} \neq \text{q, where } \lambda \in (0,1),$$

in the sense that $d(T\text{p}, T\text{q})$ lies strictly inside the solid cone with respect to $d(\text{p,q})$. Assume additionally that T is continuous. Then T has a unique fixed point.

Proof: Step 1: Construction of Iterates. Let $x_0 \in X$and $x_{s+1} = Tx_s$. We claim $\{x_s\}$ is Cauchy. Fix $c \in \text{int}(P)$. Since P is solid, for any $\delta > 0$, the set $B(c, \delta) \cap E$ intersects $\text{int}(P)$

By the contractive hypothesis and induction: $d(x_{s+1}, x_s) \prec \lambda^s d(x_1, x_0)$.

For any $c \in \text{int}(P)$, since $\lambda^s \to 0$, there exists N_0 such that $\lambda^s d(x_1, x_0) \prec c(1 - \lambda)$ for all s $\geq N_0$. Then for $m > \text{s} \geq N_0$:

$$d(x_m, x_s) \prec \frac{\lambda^s}{1 - \lambda} d(x_1, x_0) \prec c$$

confirming the Cauchy property in the non-normal cone sense (Lemma 2).

By completeness $x_s \to \zeta$ for some $\zeta \in X$. Continuity of T gives $T\zeta = \zeta$.

If $T\zeta = \zeta$ and $T\eta = \eta$ with $\zeta \neq \eta$, then $d(\zeta, \eta) = d(T\zeta, T\eta) \prec \lambda d(\zeta, \eta)$, giving $(1 - \lambda)d(\zeta, \eta) \prec 0$. Since $d(\zeta, \eta) \in P$ and $1 - \lambda > 0$, this implies $d(\zeta, \eta) \preceq 0$, so

$d(\zeta, \eta) = 0$, i.e., $\zeta = \eta$.

Theorem 4. (Hardy-Rogers Type in Non-Normal Cone Spaces). Let (X, d) be a complete cone metric space over a solid cone P. Suppose $T: X \to X$ satisfies:

$$d(Tp, Tq) \preceq a_1 d(p,q) + a_2 d(p, Tp) + a_3 d(q, Tq) + a_4 d(p, Tq) + a_5 d(q, Tp)$$

where $a_i \geq 0$ and $a_1 + a_2 + a_3 + 2a_4 < 1$, $a_1 + a_2 + a_3 + 2a_5 < 1$. Then T has a unique fixed point.

Proof: The proof adapts the original argument to the cone setting. The critical step involves establishing that the Picard orbit forms a Cauchy sequence using the interior-point criterion of Lemma 2, replacing norm estimates with interior-cone comparisons throughout. The condition $a_1 + a_2 + a_3 + 2a_4 < 1$ ensures the geometric series arising from the triangle inequality converges in the cone order sense.

Common Fixed Points in Cone Metric Spaces

We now turn to common fixed point theory for pairs of mappings.

Definition 4. (Compatible Mappings in Cone Metric Spaces). Two mappings $F, G: X \to X$ on a cone metric space (X, d) are called compatible if:

$$\lim_{n\to\infty} d(FGx_n, GFx_n) = 0_E$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n Fx_n = \lim_n Gx_n \in X$.

Theorem 5. (Common Fixed Point for Compatible Mappings). Let (X, d) be a complete cone metric space over a normal cone P. Let $F, G: X \to X$ be compatible mappings satisfying $G(X) \subseteq F(X)$ and, for all $x, y \in X$:

$$d(Gx, Gy) \preceq k \cdot d(Fx, Fy), k \in [0, \frac{1}{N})$$

where N is the normal constant of P. If F or G is continuous, then F and G have a unique common fixed point.

Proof: Since $\mathrm{G}(X) \subseteq \mathrm{F}(X)$, we can define a sequence $\{y_n\}$in X by choosing $y_0 \in X$

and inductively setting $\mathrm{F}y_{n+1} = \mathrm{G}y_n$. Let $z_n = \mathrm{G}y_n = \mathrm{F}y_{n+1}$.

Applying the contractive condition:

$$d(z_{n+1}, z_n) = d(\mathrm{G}y_{n+1}, \mathrm{G}y_n) \preceq k \cdot d(\mathrm{F}y_{n+1}, \mathrm{F}y_n) = k \cdot d(z_n, z_{n-1})$$

By induction, $d(z_{n+1}, z_n) \preceq k^n d(z_1, z_0)$.

Since $k < 1/N$, we have $Nk < 1$ and by normality:

$$\| d(z_m, z_n) \| \leq N \| \sum_{j=n}^{m-1} d(z_{j+1}, z_j) \| \leq N \cdot \frac{(kN)^n}{1 - kN} \| d(z_1, z_0) \| \to 0.$$

Fixed Point. By completeness, $z_n \to z^*$. Suppose F is continuous; then $\mathrm{F}z_n \to \mathrm{F}z^*$and compatibility of F, G implies $\mathrm{G}z^* = \mathrm{F}z^*$. Applying the contractive condition with $x = z^*$and any y_n:

$$d(\mathrm{G}z^*, \mathrm{G}y_n) \preceq k \cdot d(\mathrm{F}z^*, \mathrm{F}y_n) = k \cdot d(\mathrm{G}z^*, z_{n-1}) \to 0.$$

Hence $\mathrm{G}z^* = z^*$and $\mathrm{F}z^* = z^*$. Uniqueness follows analogously to Theorem 1.

Note. If in Theorem 5 we take $\mathrm{F} = \mathrm{Id}$(identity), we recover the standard Banach contraction principle in cone metric spaces. If $\mathrm{F} = \mathrm{G}$, we obtain a coincidence point theorem for a single self-mapping with a Lipschitz iterate condition.

The fixed point theory developed in above is related with single-valued mappings. However, many problems in the theory of analysis, control theory and game theory have mappings that assign to each point a set of possible images rather than one unique image. This section is the study of fixed point theory for such multivalued mappings in cone metric spaces, extending the single-valued results in a non-trivial way.

Definition 5. (Multivalued Mapping and Fixed Point). Let (X, d)be a cone metric space. A multivalued mapping is function $\mathcal{T}: X \to 2^X \setminus \{\emptyset\}$, where 2^Xdenotes the power set of X. A point $\zeta \in X$ is a fixed point of $\mathcal{T}$ if $\zeta \in \mathcal{T}(\zeta)$.

Definition 6. (Hausdorff Cone Metric). Let (X, d)be a cone metric space over cone P in E and let $CB(X)$ denote the collection of all non-empty closed bounded subsets of X. The Hausdorff cone metric G: $CB(X) \times CB(X) \to E$ is defined by:

$$\mathrm{G}(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}$$

where $D(x, B) = \inf_{y\in B} d(x, y)$is the cone distance from a point to a set, with the infimum taken in the cone order sense.

Note. The Hausdorff cone metric satisfy all cone metric properties from d: symmetry, the condition $\mathrm{G}(A, B) = 0_E \Leftrightarrow A = B$ and the triangle inequality $\mathrm{G}(A, C) \preceq \mathrm{G}(A, B) + \mathrm{G}(B, C)$. However, verifying that $D(x, B)$is well-defined in the cone order requires additional care: since the cone Pmay not have the infimum property, $D(x, B)$is defined as the greatest lower bound of $\{d(x, y): y \in B\}$in the cone order, which exists when P is normal (as order-bounded decreasing nets converge in norm).

Theorem 6. (Nadler-Type Fixed Point in Cone Metric Spaces). Let (X, d)be a complete cone metric space over a normal cone P with normal constant N. Let $\mathcal{T}: X \to CB(X)$be a multivalued mapping satisfying:

$$\mathrm{G}(\mathcal{T}\mathrm{p}, \mathcal{T}\mathrm{q}) \preceq k \cdot d(\mathrm{p},\mathrm{q}) \text{for all } \mathrm{p}, \mathrm{q} \in X,$$

where $k \in [0,1/(N + 1))$. Then $\mathcal{T}$has a fixed point.

Proof : Fix $p_0 \in X$ and choose $p_1 \in \mathcal{T}(p_0)$. Since $p_1 \in \mathcal{T}(p_0)$, we have $D(p_1, \mathcal{T}(p_1)) \preceq H(\mathcal{T}(p_0), \mathcal{T}(p_1)) \preceq k \cdot d(p_0, p_1)$.

Now, for any $c \in \text{int}(P)$ and using the definition of D, there exists $p_2 \in \mathcal{T}(p_1)$ such that:

$$d(p_1, p_2) \preceq D(p_1, \mathcal{T}(p_1)) + \frac{c}{2} \preceq k \cdot d(p_0, p_1) + \frac{c}{2}.$$

More carefully, choosing iteratively and taking $c \to 0$ in the interior-point sense yields a sequence $\{p_n\}$ with $p_{n+1} \in \mathcal{T}(p_n)$ and:

$$d(p_{n+1}, p_n) \preceq \frac{k^n}{1-k} d(p_1, p_0) \cdot (1+k)^n.$$

For the normality condition to yield summability, we require $k(N+1) < 1$, which is the stated

hypothesis. Under this condition $\{p_n\}$ is Cauchy and by completeness converges to some $\zeta \in X$.

Since each $\mathcal{T}(p_n)$ is closed and $p_{n+1} \in \mathcal{T}(p_n)$ and since

$H(\mathcal{T}(p_n), \mathcal{T}(\zeta)) \preceq k \cdot d(p_n, \zeta) \to 0_E$, we get $D(\zeta, \mathcal{T}(\zeta)) = \lim_n D(p_{n+1}, \mathcal{T}(\zeta)) \preceq$

$\lim_n H(\mathcal{T}(p_n), \mathcal{T}(\zeta)) = 0_E$. Hence $\zeta \in \mathcal{T}(\bar{\zeta}) = \mathcal{T}(\zeta)$.

Note. When $\mathcal{T}$ is assumed as single-valued (each $\mathcal{T}(p)$is a singleton), Theorem 6 reduces to Theorem 1 with $\alpha = k$ and $\beta = 0$, recovering the standard Banach fixed point in cone metric spaces. Thus multivalued theory genuinely subsumes the single-valued case.

- **Coupled Fixed Points in Cone Metric Spaces**

Coupled fixed point theory, initiated by Guo and [15] and extended to ordered metric spaces by Bhaskar and Lakshmikantham (2006), provides a powerful framework for analyzing systems of equations with order-theoretic structure. We develop this theory in the cone metric setting.

Definition 7. (Coupled Fixed Point). Let $\mathcal{F}: X \times X \to X$ be a mapping on a cone metric space

(X, d). A pair $(u, v) \in X \times X$ is called a coupled fixed point of $\mathcal{F}$ if:

$$\mathcal{F}(u, v) = u \text{ and } \mathcal{F}(v, u) = v.$$

Definition 8. (Mixed Monotone Property). A mapping $\mathcal{F}: X \times X \to X$ on a partially ordered cone metric space $(X, d, \preceq)$has the mixed monotone property if $\mathcal{F}(x, \cdot)$is decreasing and $\mathcal{F}(\cdot, y)$is increasing in the partial order: $x_1 \preceq x_2 \Rightarrow \mathcal{F}(x_1, y) \preceq \mathcal{F}(x_2, y)$ and $y_1 \preceq y_2 \Rightarrow \mathcal{F}(x, y_2) \preceq \mathcal{F}(x, y_1)$.

Theorem 7. (Coupled Fixed Point Theorem in Ordered Cone Metric Spaces). Let $(X, d, \preceq)$be a complete partially ordered cone metric space over a normal cone P with normal constant N. Let $\mathcal{F}: X \times X \to X$have the mixed monotone property and satisfy:

$$d(\mathcal{F}(x, y), \mathcal{F}(u, v)) \preceq \frac{k}{2}[d(x, u) + d(y, v)]$$

for all $x \succeq u$ and $y \preceq v$, where $k \in [0,1/N)$. If there exists $(x_0, y_0) \in X \times X$ with $x_0 \preceq \mathcal{F}(x_0, y_0)$ and $y_0 \succeq \mathcal{F}(y_0, x_0)$, then $\mathcal{F}$ has a coupled fixed

Conclusion: The Study as discussed above has developed a large theoretical framework for fixed point theory in cone metric spaces involving normal and non-normal cone metric spaces. This Paper starts with the basic definitions, structural properties and preliminary results required for working in these generalized settings. Special obeservation was given to the distinction between normal and non-normal cones, since this distinction affects directly to convergence arguments, completeness criteria and the methods available for proving fixed point results. The large contribution of the study lies in the development of new fixed point theorems for generalized contractions in cone metric spaces. In normal cone metric spaces, Kannan-type and Ćirić-type contractions were extended successfully, producing existence, uniqueness and convergence results. These results show that the cone setting can accommodate mixed contractive conditions that unify several classical contraction types within a broader order-valued setting. In non-normal cone metric spaces, where norm-based techniques are no longer sufficient, the study introduced alternative approaches based on interior-point arguments and cone-order comparisons. These methods made it possible to derive fixed point theorems under weaker structural assumptions and thereby address one of the most significant gaps in the existing literature.

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