JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

M-POLYNOMIALS AND DEGREE BASED TOPOLOGICAL INDICES OF WHEEL RELATED GRAPHS

Ms. P. Jansi Rathina Kumari [1], Dr. R. Ganapathy Raman[2]

[1] Research Scholar, PG & Research Department of Mathematics, Pachaiyappa's College, Chennai-30, India.

[2] Assistant Professor,PG & Research Department of Mathematics Pachaiyappa's College, Chennai-30, India.

Corresponding Author Email: jansikmr@gmail.com

Abstract:

The M-Polynomial[3] of a graph G is defined as $M(G;\, x, y) = \sum_{i\le j} m_{ij}(G)\, x^i y^j$ where $m_{ij}, i, j \ge 1$, is the number of edges uv of G such that $\{d_G(u), d_G(v)\} = \{i, j\}$. Topological indices, especially degree-based topological indices, which can be determined with an algebraic approach known as the M-polynomial, are utilized to determine the physiochemical characteristics of chemical graphs. For some wheel-related graphs we found M-polynomials, and thereby we have established various degree-based topological indices for those graphs.

Key words: M-Polynomials, Wheel graphs, topological indices

Subject Classification: 05C92

I. INTRODUCTION

Let $G = \big(V(G), E(G)\big)$ be a simple, connected, finite, and undirected graph. The degree $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in G. An isolated vertex or singleton graph is a vertex with degree zero. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of G and let $d_i = d_G(v_i)$. Several topological indices have been defined in the literature. Among them some degree based topological indices are First Zagreb index [7], Second Zagreb index, modified second Zagreb index, Symmetric division degree index [10], Harmonic index [4], Inverse sum index [10] and Forgotten index [6].

The general form of these degree-based topological indices of a graph is given by $TI(G) = \sum_{e=uv\in E(G)} f(d_G(u), d_G(v))$, where $f = f(x, y)$ is a function appropriately chosen for the computation. Table 1 gives some topological indices defined by $f(x, y)$. It would be interesting that, if all these topological indices are obtained from a single expression. This role is played by polynomials. In fact there are several graph polynomials like Hosoya polynomial [3], matching polynomial [5], Schultz polynomial [8], M-Polynomial [3], NM Polynomial etc., Among them, the M-polynomial is the well-known polynomial which was introduced in 2015 by Deutsch and Klavzar, which is useful in determining many degree-based topological indices (listed in Tables 1). This motivates us to study M-polynomial of some graph operations and wheel related graphs.

Table 1. [1] Operators to derive degree based topological indices from M-polynomial.

S.No.	Notation	Topological Index	*f*(*x*, *y*)	Derivation from M(G; *x*, *y*)
1	$M_1(G)$	First Zagreb Index	$x + y$	$(D_x + D_y)(M(G; x, y))\vert_{x=y=1}$
2	$M_2(G)$	Second Zagreb Index	xy	$(D_x D_y)(M(G; x, y))\vert_{x=y=1}$
3	$M_2^m(G)$	Second modified Zagreb Index	$\frac{1}{xy}$	$(S_x S_y)(M(G; x, y))\vert_{x=y=1}$
4	SDD(G)	Symmetric division degree Index	$\frac{x^2 + y^2}{xy}$	$(D_x S_y + D_y S_x)(M(G; x, y))\vert_{x=y=1}$
5	H(G)	Harmonic Index	$\frac{2}{x + y}$	$2S_x J(M(G; x, y))\vert_{x=1}$

S.No	Notation	Topological Index	f(x, y)	Derivation from M(G; x, y)
6	ISI(G)	Inverse sum Indeg Index	$\frac{xy}{x+y}$	$2S_xJD_xD_y(M(G;x,y))\|_{x=1}$
7	F(G)	Forgotten Index	x^2+y^2	$(D_xD_x+D_yD_y)(M(G;x,y))\|_{x=y=1}$

where $D_x = x\frac{\partial f(x,y)}{\partial x}$, $D_y = y\frac{\partial f(x,y)}{\partial y}$, $S_x = \int_0^x \frac{f(t,y)}{t}dt$, $S_y = \int_0^y \frac{f(x,t)}{t}dt$ and $J(f(x,y)) = f(x,x)$ are the operators

II. PRELIMINARIES

Definition:2.1 Water wheel graph [2] (WW_n)

For $n \geq 3$, a water wheel graph, $\mathrm{WW_n}$, is a graph with
(i) A vertex set $\mathrm{V}(\mathrm{WW}_n) = \{u_0\} \cup \{u_i, v_i, w_i\ ; 1 \leq i \leq n\}$
(ii) An edge set $\mathrm{E}(\mathrm{WW}_n) = \{u_0u_i, u_0v_i, u_iw_i, v_iw_i\ ; 1 \leq i \leq n\} \cup \{u_iv_{i+1}\ ; 1 \leq i \leq n-1\} \cup \{u_nv_1\}$

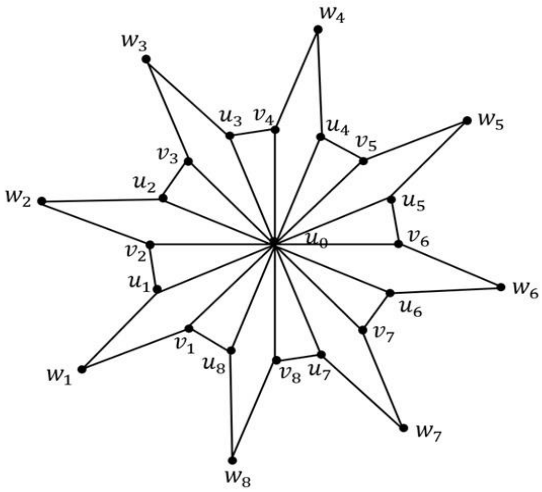

Figure 1: $\mathrm{WW_8}$

Definition: 2.2 Triangulated water wheel graph [2] (TWW_n)

For $n \geq 3$, a triangulated water wheel graph TWW_n is obtained by adding edges $\mathrm{u_0w_i}$, for all $1 \leq i \leq n$, to the WW_n graph.

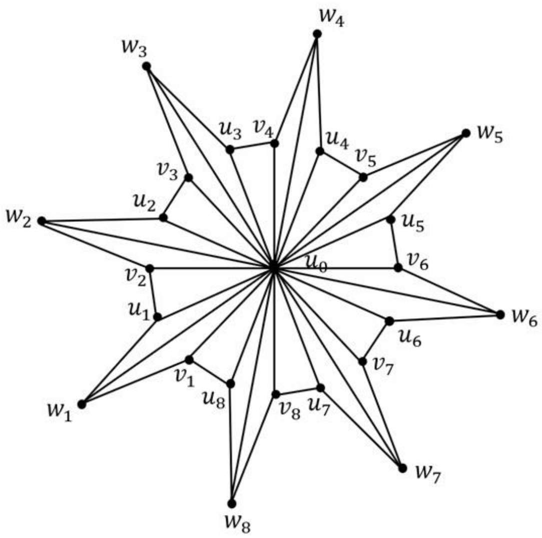

Figure 2: $\mathrm{TWW_8}$

Definition: 2.3 Closed water wheel graph [2] (CWW_n)

For $n \geq 3$, a closed water wheel graph CWW_n , is obtained by adding edges w_1w_n and w_iw_{i+1} , $1 \leq i \leq n-1$, to the WW_n graph.

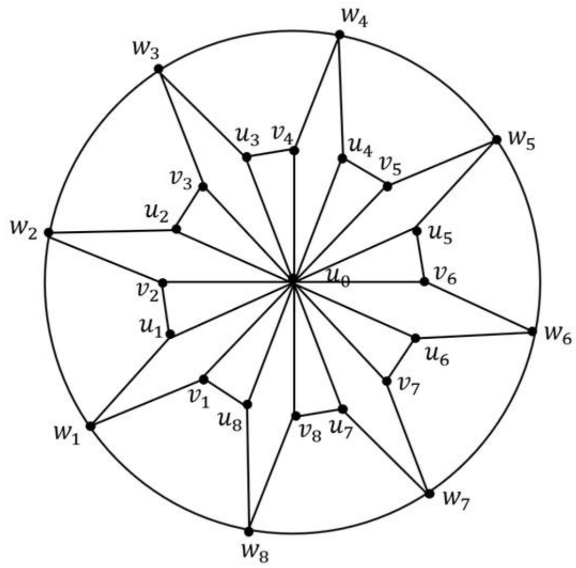

Figure 3: $\mathbf{CWW_8}$

Definition: 2.4 Closed triangulated water wheel graph [2] ($\mathbf{CTWW_n}$)

For $n \geq 3$, a closed triangulated water wheel graph CTWW_n is obtained by adding edges $w_1 w_n$ and $w_i w_{i+1}$, $1 \leq i \leq n-1$, to the TWW_n graph.

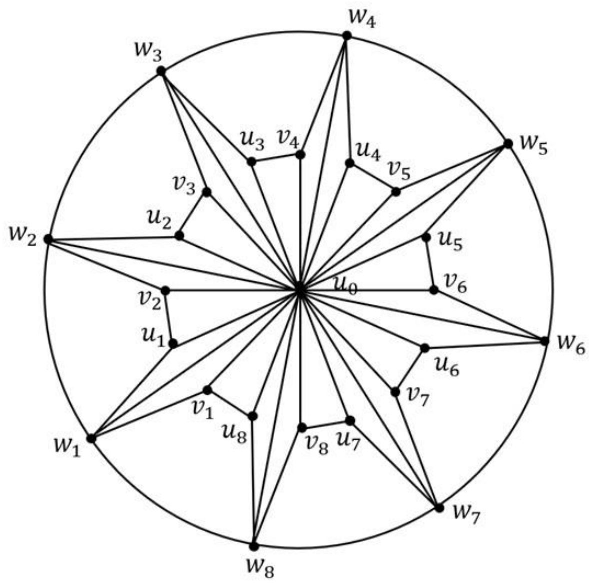

Figure 4: $\mathbf{CTWW_8}$

Definition: 2.5 Cog Wheel graph [1] ($\mathbf{CW_n}$)

It is obtained by combining a wheel graph $w_n = \{u_0, v_1, v_2, v_3, \dots, v_n\}$ with a set of vertices $\{u_1, u_2, u_3, \dots, u_n\}$ such that a vertex u_i is adjacent to vertices v_i and v_{i+1} , for $1 \leq i \leq n-1$. Furthermore, the vertex u_n is adjacent to vertices v_1 and v_n.

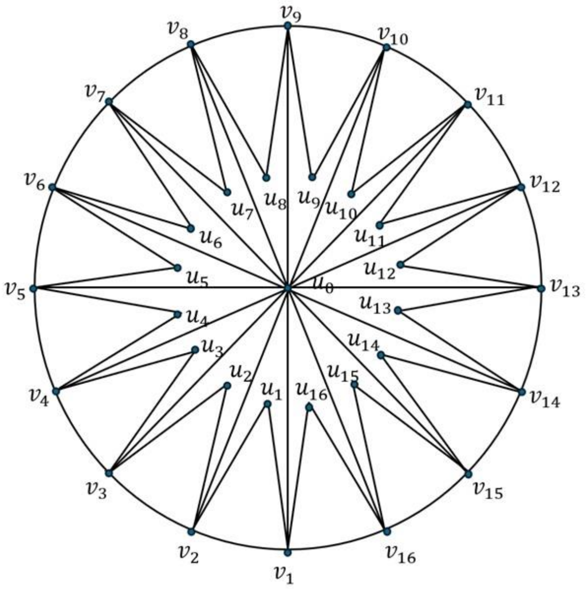

Figure 5: $\mathbf{CW_{16}}$

Definition: 2.6 Triangulated wheel graph [1] ($\mathbf{TW_n}$)

It is obtained by combining the wheel graph $w_n = \{u_0, v_1, v_2, v_3, \dots, v_n\}$ with a set of vertices $\{u_1, u_2, u_3, \dots, u_n\}$ such that a vertex u_i is adjacent to vertices v_i and v_{i+1}, for $1 \le i \le n-1$ and the vertex u_n is adjacent to vertices v_1 and v_n. Additionally, each vertex u_i is adjacent to the vertex u_0.

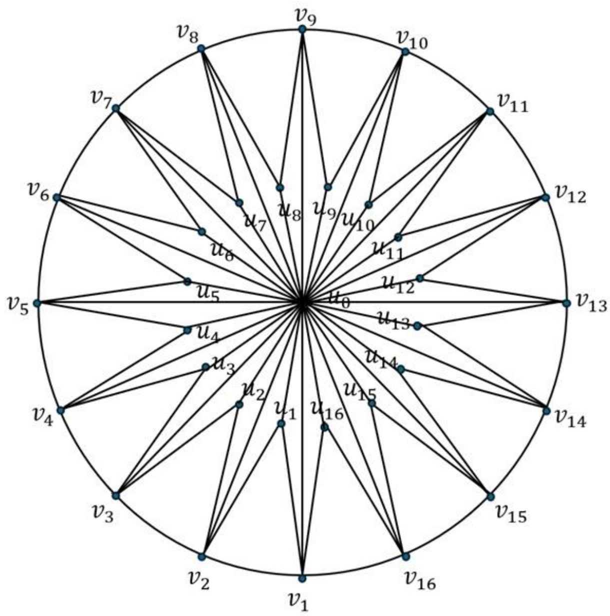

Figure 6: $\mathbf{TW_{16}}$

Definition: 2.7 Double crown wheel graph [1] ($\mathbf{DCW_n}$)

It is obtained by combining a wheel graph $w_n = \{u_0, v_1, v_2, v_3, \dots, v_n\}$ with two set of vertices. The first set is $\{w_1, w_2, w_3, \dots, w_n\}$, where each vertex w_i is adjacent to vertices v_i and v_{i+1}, for $1 \le i \le n-1$, and the vertex w_n is adjacent to vertices v_1 and v_n. The second set is $\{u_1, u_2, u_3, \dots, u_n\}$, where each vertex u_i is adjacent to vertices v_i and v_{i+1}, for $1 \le i \le n-1$, and the vertex u_n is adjacent to vertices v_1 and v_n.

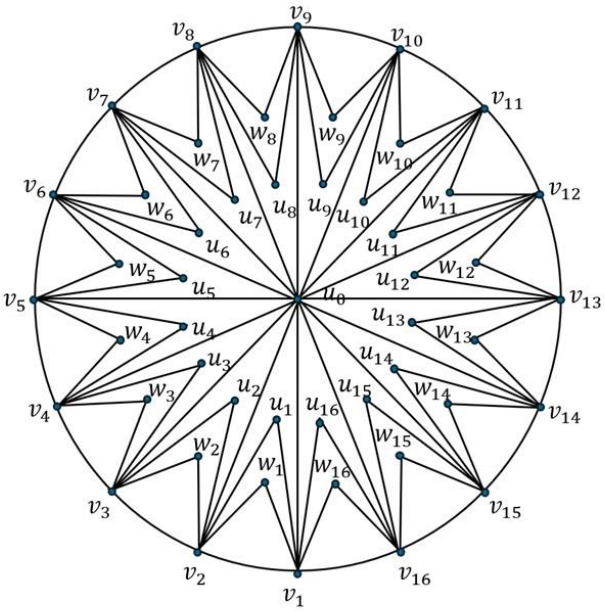

Figure 7: $\mathbf{DCW_{16}}$

Definition: 2.8 Crown triangulated wheel graph [1] $(\mathbf{CTW}_n)$

It is obtained by combining the triangulated wheel graph $TW_n = \{u_0, v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, \ldots, u_n\}$ with the set $\{w_1, w_2, w_3, \ldots, w_n\}$, where each vertex w_i is adjacent to vertices v_i and v_{i+1} for $1 \le i \le n-1$, and the vertex w_n is adjacent to vertices v_1 and v_n.

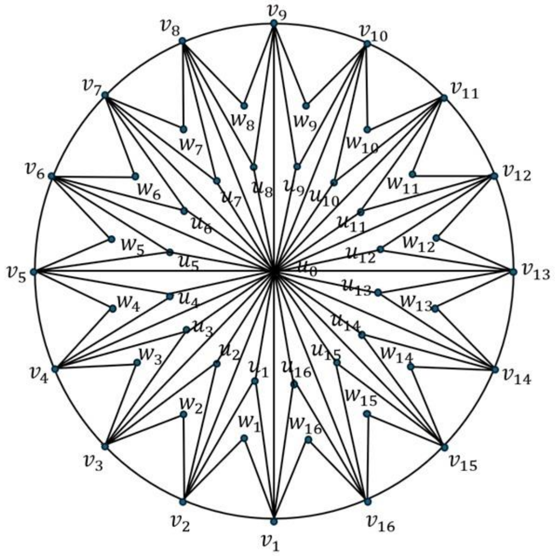

Figure 8: $\mathbf{CTW_{16}}$

Definition: 2.9 M-Polynomial [3]

Let G be a graph. Then M-Polynomial of G is defined as

$$M(G;\ x, y) = \sum_{i \le j} m_{ij}(G)\, x^i y^j$$

where $m_{ij}, i, j \ge 1$, is the number of edges uv of G such that $\{d_G(u), d_G(v)\} = \{i, j\}$.

III. MAIN RESULTS

Theorem: 3.1

Let $\mathrm{WW_n}$ be a water wheel graph, then the M-polynomial of a Water wheel graph is $M(WW_n; x, y) = 2nx^3y^{2n} + nx^3y^3 + 2nx^2y^3$.

Proof:
The Water wheel graph WW_n contains 3n+1vertices and 5n edges.

We divide the vertex set V of WW_n into four disjoint parts according to the degree of vertices as: $d(u_0) = 2n$, $d(u_i) = 3$, $d(v_i) = 3$ and $d(w_i) = 2$, for all i=1, 2, …, n
For $n \geq 3$, the edge partition is $|E_{\{d(u),d(v)\}}| = |E_{\{2,3\}}| = 2n$, $|E_{\{3,3\}}| = n$, $|E_{\{3,2n\}}| = 2n$.

Then M-Polynomial of water wheel graph $\mathrm{WW_n}$ is defined by

$$M(\mathrm{WW_n};\ x,y) = \sum_{i\leq j} m_{ij}(G)\, x^i y^j$$

$$M(WW_n;\ x,y) = \sum_{3\leq 2n} m_{3,2n}(G)\, x^3 y^{2n} + \sum_{3\leq 3} m_{3,3}(G)\, x^3 y^3 + \sum_{2\leq 3} m_{2,3}(G)\, x^2 y^3$$
$$= |E_{\{3,2n\}}| x^3 y^{2n} + |E_{\{3,3\}}| x^3 y^3 + |E_{\{2,3\}}| x^2 y^3$$
$$M(WW_n; x,y) = 2nx^3y^{2n} + nx^3y^3 + 2nx^2y^3.$$

Corollary: 3.1

If WW_n is a Water wheel graph, then

(i) $M_1(WW_n) = 4n^2 + 22n$
(ii) $M_2(WW_n) = 12n^2 + 21n$
(iii) $M_2^m(WW_n) = \frac{4n+3}{9}$
(iv) $SDD(WW_n) = \frac{4n^2+19n+9}{3}$
(v) $H(WW_n) = \frac{17n}{15} + \frac{4n}{2n+3}$
(vi) $ISI(WW_n) = \frac{39n}{10} + \frac{12n^2}{2n+3}$
(vii) $F(WW_n) = 8n^3 + 62n$

Proof:
The M-polynomial of a Water wheel graph WW_n is given by
$M(WW_n; x,y) = 2nx^3y^{2n} + nx^3y^3 + 2nx^2y^3$.
Using the expressions from Table 1, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 6nx^3y^{2n} + 3nx^3y^3 + 4nx^2y^3$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = 4n^2x^3y^{2n} + 3nx^3y^3 + 6nx^2y^3$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = \frac{2nx^3y^{2n}}{3} + \frac{nx^3y^3}{3} + nx^2y^3$$
$$S_y = \int_0^y \frac{f(x,t)}{t}dt = x^3y^{2n} + \frac{nx^3y^3}{3} + \frac{2nx^2y^3}{3}$$

Therefore,

$$M_1(WW_n) = (D_x + D_y)(M(WW_n; x,y))|_{x=y=1} = 4n^2 + 22n$$
$$M_2(WW_n) = (D_xD_y)(M(WW_n; x,y))|_{x=y=1} = 12n^2 + 21n$$
$$M_2^m(WW_n) = (S_xS_y)(M(WW_n; x,y))|_{x=y=1} = \frac{4n+3}{9}$$
$$SDD(WW_n) = (D_xS_y + D_yS_x)(M(WW_n; x,y))|_{x=y=1} = \frac{4n^2+19n+9}{3}$$
$$H(WW_n) = 2S_xJ(M(WW_n; x,y))|_{x=1} = \frac{17n}{15} + \frac{4n}{2n+3}$$
$$ISI(WW_n) = 2S_xJD_xD_y(M(WW_n; x,y))|_{x=1} = \frac{39n}{10} + \frac{12n^2}{2n+3}$$
$$F(WW_n) = (D_xD_x + D_yD_y)(M(WW_n; x,y))|_{x=y=1} = 8n^3 + 62n.$$

Theorem: 3.2

Let $T\mathrm{WW_n}$ be a Triangulated water wheel graph, then the M-polynomial of a Triangulated water wheel graph is $M(TWW_n; x,y) = 3nx^3y^{3n} + 3nx^3y^3$.

Proof:
The Triangulated Water wheel graph TWW_n contains $3n$+1vertices and $6n$ edges.
We divide the vertex set V of $T\mathrm{WW_n}$ into four disjoint parts according to the degree of vertices as: $d(u_0) = 3n$, $d(u_i) = 3$, $d(v_i) = 3$ and $d(w_i) = 3$, for all i=1, 2, …, n
For $n \geq 3$, the edge partition is $|E_{\{3,3\}}| = 3n$, $|E_{\{3,3n\}}| = 3n$.

Then M-Polynomial of Triangulated water wheel graph $T\mathrm{WW_n}$ is defined by

$$M(TWW_n;\ x,y) = \sum_{i\leq j} m_{ij}(G)\, x^i y^j$$

$$M(TWW_n;\ x,y) = \sum_{3\le 3n} m_{3,3n}(G)\,x^3y^{3n} + \sum_{3\le 3} m_{3,3}(G)\,x^3y^3$$
$$= |E_{\{3,3n\}}|x^3y^{3n} + |E_{\{3,3\}}|x^3y^3$$
$$M(TWW_n;x,y) = 3nx^3y^{3n} + 3nx^3y^3.$$

Corollary: 3.2

If TWW_n is a Triangulated Water wheel graph, then

(i) $M_1(TWW_n) = 9n^2 + 27n$
(ii) $M_2(TWW_n) = 27n^2 + 27n$
(iii) $M_2^m(TWW_n) = \frac{n+1}{3}$
(iv) $SDD(TWW_n) = 3n^2 + 6n + 3$
(v) $H(TWW_n) = \frac{n^2+3n}{n+1}$
(vi) $ISI(TWW_n) = \frac{27n^2+9n}{2(n+1)}$
(vii) $F(TWW_n) = 27n^3 + 81n$

Proof:
The M-polynomial of a Triangulated Water wheel graph TWW_n is given by
$M(TWW_n;x,y) = 3nx^3y^{3n} + 3nx^3y^3$.

Using the expressions from Table 1, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 9nx^3y^{3n} + 9nx^3y^3$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = 9n^2x^3y^{3n} + 9nx^3y^3$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = nx^3y^{3n} + nx^3y^3$$
$$S_y = \int_0^y \frac{f(x,t)}{t}dt = x^3y^{3n} + nx^3y^3$$

Therefore,

$$M_1(TWW_n) = (D_x + D_y)(M(TWW_n;x,y))|_{x=y=1} = 9n^2 + 27n$$
$$M_2(TWW_n) = (D_xD_y)(M(TWW_n;x,y))|_{x=y=1} = 27n^2 + 27n$$
$$M_2^m(TWW_n) = (S_xS_y)(M(TWW_n;x,y))|_{x=y=1} = \frac{n+1}{3}$$
$$SDD(TWW_n) = (D_xS_y + D_yS_x)(M(TWW_n;x,y))|_{x=y=1} = 3n^2 + 6n + 3$$
$$H(TWW_n) = 2S_xJ(M(TWW_n;x,y))|_{x=1} = \frac{n^2+3n}{n+1}$$
$$ISI(TWW_n) = 2S_xJD_xD_y(M(TWW_n;x,y))|_{x=1} = \frac{27n^2+9n}{2(n+1)}$$
$$F(TWW_n) = (D_xD_x + D_yD_y)(M(TWW_n;x,y))|_{x=y=1} = 27n^3 + 81n.$$

Theorem:3.3

Let CWW_n be a Closed water wheel graph, then the M-polynomial of a Closed water wheel graph is $M(CWW_n;x,y) = 2nx^3y^{2n} + nx^4y^4 + 2nx^3y^4 + nx^3y^3$.

Proof:
The Closed water wheel graph CWW_n contains 3n+1vertices and 6n edges.
We divide the vertex set V of CWW_n into four disjoint parts according to the degree of vertices as: $d(u_0) = 2n$, $d(u_i) = 3$, $d(v_i) = 3$ and $d(w_i) = 4$, for all i=1, 2, …, n
For $n \ge 3$, the edge partition is $|E_{\{3,3\}}| = n$, $|E_{\{3,4\}}| = 2n$, $|E_{\{4,4\}}| = n$, $|E_{\{3,2n\}}| = 2n$.

Then M-Polynomial of Closed water wheel graph CWW_n is defined by

$$M(CWW_n;\ x,y) = \sum_{i\le j} m_{ij}(G)\,x^iy^j$$

$$M(CWW_n;\ x,y) = \sum_{3\le 2n} m_{3,2n}(G)\,x^3y^{2n} + \sum_{4\le 4} m_{4,4}(G)\,x^4y^4 + \sum_{3\le 4} m_{3,4}(G)\,x^3y^4 + \sum_{3\le 3} m_{3,3}(G)\,x^3y^3$$
$$= |E_{\{3,2n\}}|x^3y^{2n} + |E_{\{4,4\}}|x^4y^4 + |E_{\{3,4\}}|x^3y^4 + |E_{\{3,3\}}|x^3y^3$$
$$M(CWW_n;\ x,y) = 2nx^3y^{2n} + nx^4y^4 + 2nx^3y^4 + nx^3y^3.$$

Corollary:3.3

If CWW_n is a Closed water wheel graph, then

(i) $M_1(CWW_n) = 4n^2 + 34n$
(ii) $M_2(CWW_n) = 12n^2 + 49n$
(iii) $M_2^m(CWW_n) = \frac{49n+48}{144}$
(iv) $SDD(CWW_n) = \frac{8n^2+49n+18}{6}$
(v) $H(CWW_n) = \frac{4n}{2n+3} + \frac{97n}{84}$
(vi) $ISI(CWW_n) = \frac{12n^2}{2n+3} + \frac{97n}{14}$
(vii) $F(CWW_n) = 8n^3 + 118n$

Proof:
The M-polynomial of a Closed water wheel graph CWW_n is given by
$M(CWW_n;\, x,y) = 2nx^3y^{2n} + nx^4y^4 + 2nx^3y^4 + nx^3y^3$.

Using the expressions from Table 1, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 6nx^3y^{2n} + 4nx^4y^4 + 6nx^3y^4 + 3nx^3y^3$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = 4n^2x^3y^{2n} + 4nx^4y^4 + 8nx^3y^4 + 3nx^3y^3$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = \frac{2nx^3y^{2n}}{3} + \frac{nx^4y^4}{4} + \frac{2nx^3y^4}{3} + \frac{nx^3y^3}{3}$$
$$S_y = \int_0^y \frac{f(x,t)}{t}dt = x^3y^{2n} + \frac{nx^4y^4}{4} + \frac{nx^3y^4}{2} + \frac{nx^3y^3}{3}$$

Therefore,

$$M_1(CWW_n) = (D_x + D_y)(M(CWW_n;x,y))|_{x=y=1} = 4n^2 + 34n$$
$$M_2(CWW_n) = (D_xD_y)(M(CWW_n;x,y))|_{x=y=1} = 12n^2 + 49n$$
$$M_2^m(CWW_n) = (S_xS_y)(M(CWW_n;x,y))|_{x=y=1} = \frac{49n+48}{144}$$
$$SDD(CWW_n) = (D_xS_y + D_yS_x)(M(CWW_n;x,y))|_{x=y=1} = \frac{8n^2+49n+18}{6}$$
$$H(CWW_n) = 2S_xJ(M(CWW_n;x,y))|_{x=1} = \frac{4n}{2n+3} + \frac{97n}{84}$$
$$ISI(CWW_n) = 2S_xJD_xD_y(M(CWW_n;x,y))|_{x=1} = \frac{12n^2}{2n+3} + \frac{97n}{14}$$
$$F(CWW_n) = (D_xD_x + D_yD_y)(M(CWW_n;x,y))|_{x=y=1} = 8n^3 + 118n.$$

Theorem: 3.4

Let $CTWW_n$ be a Closed triangulated water wheel graph, then the M-polynomial of a Closed triangulated water wheel graph is $M(CTWW_n;x,y) = nx^5y^{3n} + 2nx^3y^{3n} + nx^5y^5 + 2nx^3y^5 + nx^3y^3$.

Proof:
The Closed triangulated water wheel graph $CTWW_n$ contains 3n+1vertices and 7n edges. We divide the vertex set V of $CTWW_n$ into four disjoint parts according to the degree of vertices as: $d(u_0) = 3n$, $d(u_i) = 3$, $d(v_i) = 3$ and $d(w_i) = 5$, for all i=1, 2, …, n
For $n \geq 3$, the edge partition is $|E_{\{3,3\}}| = n$, $|E_{\{3,5\}}| = 2n$, $|E_{\{5,5\}}| = n$, $|E_{\{3,3n\}}| = 2n$, $|E_{\{5,3n\}}| = n$.
Then M-Polynomial of Closed triangulated water wheel graph $CTWW_n$ is defined by

$$M(CTWW_n;\, x,y) = \sum_{i\leq j} m_{ij}(G)\, x^iy^j$$
$$M(CTWW_n;\, x,y) = \sum_{5\leq 3n} m_{5,3n}(G)\, x^5y^{3n} + \sum_{3\leq 3n} m_{3,3n}(G)\, x^3y^{3n} + \sum_{5\leq 5} m_{5,5}(G)\, x^5y^5$$
$$+ \sum_{3\leq 5} m_{3,5}(G)\, x^3y^5 + \sum_{3\leq 3} m_{3,3}(G)\, x^3y^3$$
$$= |E_{\{5,3n\}}|x^5y^{3n} + |E_{\{3,3n\}}|x^3y^{3n} + |E_{\{5,5\}}|x^5y^5 + |E_{\{3,5\}}|x^3y^5 + |E_{\{3,3\}}|x^3y^3.$$
$$M(CTWW_n;\, x,y) = nx^5y^{3n} + 2nx^3y^{3n} + nx^5y^5 + 2nx^3y^5 + nx^3y^3.$$

Corollary:3.4

If $CTWW_n$ is a Closed triangulated water wheel graph, then

(i) $M_1(CTWW_n) = 9n^2 + 43n$
(ii) $M_2(CTWW_n) = 33n^2 + 64n$

(iii) $M_2^m(CTWW_n) = \frac{64n+65}{225}$
(iv) $SDD(CTWW_n) = \frac{39n^2+128n+55}{15}$
(v) $H(CTWW_n) = \frac{18n^2+26n}{9n^2+24n+15} + \frac{31n}{30}$
(vi) $ISI(CTWW_n) = \frac{99n^3+135n^2}{9n^2+24n+15} + \frac{31n}{4}$
(vii) $F(CTWW_n) = 27n^3 + 179n$

Proof:
The M-polynomial of a Closed triangulated water wheel graph $CTWW_n$ is given by
$M(CTWW_n;\ x,y) = nx^5y^{3n} + 2nx^3y^{3n} + nx^5y^5 + 2nx^3y^5 + nx^3y^3$.
Using the expressions from Table 1, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 5nx^5y^{3n} + 6nx^3y^{3n} + 5nx^5y^5 + 6nx^3y^5 + 3nx^3y^3$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = 3n^2x^5y^{3n} + 6n^2x^3y^{3n} + 5nx^5y^5 + 10nx^3y^5 + 3nx^3y^3$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = \frac{nx^5y^{3n}}{5} + \frac{2nx^3y^{3n}}{3} + \frac{nx^5y^5}{5} + \frac{2nx^3y^5}{3} + \frac{nx^3y^3}{3}$$
$$S_y = \int_0^y \frac{f(x,t)}{t}dt = \frac{x^5y^{3n}}{3} + \frac{2x^3y^{3n}}{3} + \frac{nx^5y^5}{5} + \frac{2nx^3y^5}{5} + \frac{nx^3y^3}{3}$$
Therefore,

$$M_1(CTWW_n) = (D_x + D_y)(M(CTWW_n;x,y))|_{x=y=1} = 9n^2 + 43n$$
$$M_2(CTWW_n) = (D_xD_y)(M(CTWW_n;x,y))|_{x=y=1} = 33n^2 + 64n$$
$$M_2^m(CTWW_n) = (S_xS_y)(M(CTWW_n;x,y))|_{x=y=1} = \frac{64n+65}{225}$$
$$SDD(CTWW_n) = (D_xS_y + D_yS_x)(M(CTWW_n;x,y))|_{x=y=1} = \frac{39n^2+128n+55}{15}$$
$$H(CTWW_n) = 2S_xJ(M(CTWW_n;x,y))|_{x=1} = \frac{18n^2+26n}{9n^2+24n+15} + \frac{31n}{30}$$
$$ISI(CTWW_n) = 2S_xJD_xD_y(M(CTWW_n;x,y))|_{x=1} = \frac{99n^3+135n^2}{9n^2+24n+15} + \frac{31n}{4}$$
$$F(CTWW_n) = (D_xD_x + D_yD_y)(M(CTWW_n;x,y))|_{x=y=1} = 27n^3 + 179n.$$

Theorem: 3.5

Let CW_n be a Cog wheel graph, then the M-polynomial of a Cog wheel graph is $M(CW_n;x,y) = nx^5y^n + nx^5y^5 + 2nx^2y^5$.

Proof:
The Cog wheel graph CW_n contains 2n+1vertices and 4n edges. We divide the vertex set V of CW_n into four disjoint parts according to the degree of vertices as:$d(u_0) = n$,$d(u_i) = 2$, and $d(v_i) = 5$, for $1 \le i \le n-1$.
The edge partition is $|E_{\{2,5\}}| = 2n$, $|E_{\{5,5\}}| = n$, $|E_{\{5,n\}}| = n$.

Then M-Polynomial of Cog wheel graph CW_n is defined by
$$M(CW_n;\ x,y) = \sum_{i\le j} m_{ij}(G)\,x^iy^j$$
$$M(CW_n;\ x,y) = \sum_{5\le n} m_{5,n}(G)\,x^5y^n + \sum_{5\le 5} m_{5,5}(G)\,x^5y^5 + \sum_{2\le 5} m_{2,5}(G)\,x^2y^5$$
$$= |E_{\{5,n\}}|x^5y^n + |E_{\{5,5\}}|x^5y^5 + |E_{\{2,5\}}|x^2y^5$$
$$M(CW_n;x,y) = nx^5y^n + nx^5y^5 + 2nx^2y^5.$$

Corollary:3.5

If CW_n is a Cog wheel graph, then
(i) $M_1(CW_n) = n^2 + 29n$
(ii) $M_2(CW_n) = 5n^2 + 45n$
(iii) $M_2^m(CW_n) = \frac{6n+5}{25}$
(iv) $SDD(CW_n) = \frac{n^2+39n+25}{5}$
(v) $H(CW_n) = \frac{27n^2+205n}{35(n+5)}$
(vi) $ISI(CW_n) = \frac{145n^2+375n}{14(n+5)}$
(vii) $F(CW_n) = n^3 + 133n$.

Proof:
The M-polynomial of a Cog wheel graph CW_n is given by
$M(CW_n;x,y) = nx^5y^n + nx^5y^5 + 2nx^2y^5$.

Using the expressions from Table 1, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 5nx^5y^n + 5nx^5y^5 + 4nx^2y^5$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = n^2x^5y^n + 5nx^5y^5 + 10nx^2y^5$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = \frac{nx^5y^n}{5} + \frac{nx^5y^5}{5} + nx^2y^5$$
$$S_y = \int_0^y \frac{f(x,t)}{t}dt = x^5y^n + \frac{nx^5y^5}{5} + \frac{2nx^2y^5}{5}$$

Therefore,

$$M_1(CW_n) = (D_x + D_y)(M(CW_n;x,y))|_{x=y=1} = n^2 + 29n$$
$$M_2(CW_n) = (D_xD_y)(M(CW_n;x,y))|_{x=y=1} = 5n^2 + 45n$$
$$M_2^m(CW_n) = (S_xS_y)(M(CW_n;x,y))|_{x=y=1} = \frac{6n+5}{25}$$
$$SDD(CW_n) = (D_xS_y + D_yS_x)(M(CW_n;x,y))|_{x=y=1} = \frac{n^2+39n+25}{5}$$
$$H(CW_n) = 2S_xJ(M(CW_n;x,y))|_{x=1} = \frac{27n^2+205n}{35(n+5)}$$
$$ISI(CW_n) = 2S_xJD_xD_y(M(CW_n;x,y))|_{x=1} = \frac{145n^2+375n}{14(n+5)}$$
$$F(CW_n) = (D_xD_x + D_yD_y)(M(CW_n;x,y))|_{x=y=1} = n^3 + 133n.$$

Theorem: 3.6

Let TW_n be a Triangulated wheel graph, then the M-polynomial of a Triangulated wheel graph is $M(TW_n;x,y) = nx^5y^{2n} + nx^3y^{2n} + nx^5y^5 + 2nx^3y^5$.

Proof:

The Triangulated wheel graph TW_n contains $2n+1$vertices and $5n$ edges. We divide the vertex set V of TW_n into four disjoint parts according to the degree of vertices as: $d(u_0) = 2n$, $d(u_i) = 3$, and $d(v_i) = 5$, for $1 \le i \le n-1$.
The edge partition is $|E_{\{3,5\}}| = 2n$, $|E_{\{5,5\}}| = n$, $|E_{\{3,2n\}}| = n$, $|E_{\{5,2n\}}| = n$.

Then M-Polynomial of Triangulated wheel graph TW_n is defined by

$$M(TW_n;\ x,y) = \sum_{i\le j} m_{ij}(G)\,x^iy^j$$

$$M(TW_n;\ x,y) = \sum_{5\le 2n} m_{5,2n}(G)\,x^5y^{2n} + \sum_{3\le 2n} m_{3,2n}(G)\,x^3y^{2n} + \sum_{5\le 5} m_{5,5}(G)\,x^5y^5 + \sum_{3\le 5} m_{3,5}(G)\,x^3y^5$$
$$= |E_{\{5,2n\}}|x^5y^{2n} + |E_{\{3,2n\}}|x^3y^{2n} + |E_{\{5,5\}}|x^5y^5 + |E_{\{3,5\}}|x^3y^5$$
$$M(TW_n;x,y) = nx^5y^{2n} + nx^3y^{2n} + nx^5y^5 + 2nx^3y^5.$$

Corollary:3.6

If TW_n is a Triangulated wheel graph, then

(i) $M_1(TW_n) = 4n^2 + 34n$
(ii) $M_2(TW_n) = 16n^2 + 55n$
(iii) $M_2^m(TW_n) = \frac{13n+20}{75}$
(iv) $SDD(TW_n) = \frac{16n^2+98n+60}{15}$
(v) $H(TW_n) = \frac{8n^2+16n}{4n^2+16n+15} + \frac{7n}{10}$
(vi) $ISI(TW_n) = \frac{32n^3+60n^2}{4n^2+16n+15} + \frac{25n}{4}$
(vii) $F(TW_n) = 8n^3 + 152n.$

Proof:

The M-polynomial of a Triangulated wheel graph TW_n is given by
$M(TW_n;x,y) = nx^5y^{2n} + nx^3y^{2n} + nx^5y^5 + 2nx^3y^5$.

Using the expressions from Table 1, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 5nx^5y^{2n} + 3nx^3y^{2n} + 5nx^5y^5 + 6nx^3y^5$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = 2n^2x^5y^{2n} + 2n^2x^3y^{2n} + 5nx^5y^5 + 10nx^3y^5$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = \frac{nx^5y^{2n}}{5} + \frac{nx^3y^{2n}}{3} + \frac{nx^5y^5}{5} + \frac{2nx^3y^5}{3}$$

$$S_y = \int_0^y \frac{f(x,t)}{t}dt = \frac{x^5y^{2n}}{2} + \frac{x^3y^{2n}}{2} + \frac{nx^5y^5}{5} + \frac{2nx^3y^5}{5}$$

Therefore,

$$M_1(TW_n) = (D_x + D_y)(M(TW_n; x,y))|_{x=y=1} = 4n^2 + 34n$$
$$M_2(TW_n) = (D_xD_y)(M(TW_n; x,y))|_{x=y=1} = 16n^2 + 55n$$
$$M_2^m(TW_n) = (S_xS_y)(M(TW_n; x,y))|_{x=y=1} = \frac{13n+20}{75}$$
$$SDD(TW_n) = (D_xS_y + D_yS_x)(M(TW_n; x,y))|_{x=y=1} = \frac{16n^2+98n+60}{15}$$
$$H(TW_n) = 2S_xJ(M(TW_n; x,y))|_{x=1} = \frac{8n^2+16n}{4n^2+16n+15} + \frac{7n}{10}$$
$$ISI(TW_n) = 2S_xJD_xD_y(M(TW_n; x,y))|_{x=1} = \frac{32n^3+60n^2}{4n^2+16n+15} + \frac{25n}{4}$$
$$F(TW_n) = (D_xD_x + D_yD_y)(M(TW_n; x,y))|_{x=y=1} = 8n^3 + 152n.$$

Theorem: 3.7

Let DCW_n be a Double crown wheel graph, then the M-polynomial of a Double crown wheel graph is $M(DCW_n; x,y) = nx^7y^n + nx^7y^7 + 4nx^2y^7$.

Proof:
The Double crown wheel graph DCW_n contains $3n$+1vertices and $6n$ edges. We divide the vertex set V of DCW_n into four disjoint parts according to the degree of vertices as:
$d(u_0) = n$, $d(u_i) = 2$, $d(v_i) = 7$ and $d(w_i) = 2$ for $1 \le i \le n-1$.
The edge partition is $|E_{\{2,7\}}| = 4n$, $|E_{\{7,7\}}| = n$, $|E_{\{7,n\}}| = n$.

Then M-Polynomial of Double crown wheel graph DCW_n is defined by

$$M(DCW_n;\ x,y) = \sum_{i\le j} m_{ij}(G)\, x^iy^j$$

$$M(DCW_n;\ x,y) = \sum_{7\le n} m_{7,n}(G)\, x^7y^n + \sum_{7\le 7} m_{7,7}(G)\, x^7y^7 + \sum_{2\le 7} m_{2,7}(G)\, x^2y^7$$
$$= |E_{\{7,n\}}|x^7y^n + |E_{\{7,7\}}|x^7y^7 + |E_{\{2,7\}}|x^2y^7$$
$$M(DCW_n; x,y) = nx^7y^n + nx^7y^7 + 4nx^2y^7.$$

Corollary:3.7

If DCW_n is a Double crown wheel graph, then

(i) $M_1(DCW_n) = n^2 + 57n$
(ii) $M_2(DCW_n) = 7n^2 + 105n$
(iii) $M_2^m(DCW_n) = \frac{15n+7}{49}$
(iv) $SDD(DCW_n) = \frac{n^2+120n+49}{7}$
(v) $H(DCW_n) = \frac{65n^2+581n}{63(n+7)}$
(vi) $ISI(DCW_n) = \frac{7n^2}{(n+7)} + \frac{175n}{18}$
(vii) $F(DCW_n) = n^3 + 359n$.

Proof:
The M-polynomial of a Double crown wheel graph DCW_n is given by
$M(DCW_n; x,y) = nx^7y^n + nx^7y^7 + 4nx^2y^7$.

Using the expressions from Table 1, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 7nx^7y^n + 7nx^7y^7 + 8nx^2y^7$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = n^2x^7y^n + 7nx^7y^7 + 28nx^2y^7$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = \frac{nx^7y^n}{7} + \frac{nx^7y^7}{7} + 2nx^2y^7.$$
$$S_y = \int_0^y \frac{f(x,t)}{t}dt = x^7y^n + \frac{nx^7y^7}{7} + \frac{4nx^2y^7}{7}$$

Therefore,

$$M_1(DCW_n) = (D_x + D_y)(M(DCW_n; x,y))|_{x=y=1} = n^2 + 57n$$
$$M_2(DCW_n) = (D_xD_y)(M(DCW_n; x,y))|_{x=y=1} = 7n^2 + 105n$$

$M_2^m(DCW_n) = (S_xS_y)(M(DCW_n;x,y))|_{x=y=1} = \frac{15n+7}{49}$

$SDD(DCW_n) = (D_xS_y + D_yS_x)(M(DCW_n;x,y))|_{x=y=1} = \frac{n^2+120n+49}{7}$

$H(DCW_n) = 2S_xJ(M(DCW_n;x,y))|_{x=1} = \frac{65n^2+581n}{63(n+7)}$

$ISI(DCW_n) = 2S_xJD_xD_y(M(DCW_n;x,y))|_{x=1} = \frac{32n^3+60n^2}{4n^2+16n+15} + \frac{25n}{4}$

$F(DCW_n) = (D_xD_x + D_yD_y)(M(DCW_n;x,y))|_{x=y=1} = 8n^3 + 152n.$

Theorem: 3.8

Let CTW_n be a Crown triangulated wheel graph, then the M-polynomial of a Crown triangulated wheel graph is $M(CTW_n;x,y) = nx^7y^{2n} + nx^3y^{2n} + nx^7y^7 + 2nx^3y^7 + 2nx^2y^7$.

Proof:

The Crown triangulated wheel graph CTW_n contains $3n$+1vertices and $6n$ edges. We divide the vertex set V of CTW_n into four disjoint parts according to the degree of vertices as:
$d(u_0) = 2n$, $d(u_i) = 3$, $d(v_i) = 7$ and $d(w_i) = 2$ for $1 \le i \le n-1$.
The edge partition is $|E_{\{2,7\}}| = 2n$, $|E_{\{3,7\}}| = 2n$, $|E_{\{7,7\}}| = n$, $|E_{\{3,2n\}}| = n$, $|E_{\{7,2n\}}| = n$.
Then M-Polynomial of Crown triangulated wheel graph CTW_n is defined by

$$M(CTW_n;\ x,y) = \sum_{i\le j} m_{ij}(G)\, x^i y^j$$

$$M(CTW_n;\ x,y) = \sum_{7\le 2n} m_{7,2n}(G)\,x^7y^{2n} + \sum_{3\le 2n} m_{3,2n}(G)\,x^3y^{2n} + \sum_{7\le 7} m_{7,7}(G)\,x^7y^7 + \sum_{3\le 7} m_{3,7}(G)\,x^3y^7 + \sum_{2\le 7} m_{2,7}(G)\,x^2y^7$$

$$= |E_{\{7,2n\}}|x^7y^{2n} + |E_{\{3,2n\}}|x^3y^{2n} + |E_{\{7,7\}}|x^7y^7 + |E_{\{2,7\}}|x^2y^7 + |E_{\{2,7\}}|x^2y^7$$

$M(CTW_n;x,y) = nx^7y^{2n} + nx^3y^{2n} + nx^7y^7 + 2nx^3y^7 + 2nx^2y^7.$

Corollary:3.8

If CTW_n is a Crown triangulated wheel graph, then

(i) $M_1(CTW_n) = 4n^2 + 62n$
(ii) $M_2(CTW_n) = 20n^2 + 119n$
(iii) $M_2^m(CTW_n) = \frac{38n+35}{147}$
(iv) $SDD(CTW_n) = \frac{20n^2+317n+105}{21}$
(v) $H(CTW_n) = \frac{8n^2+20n}{4n^2+20n+21} + \frac{311n}{315}$
(vi) $ISI(CTW_n) = \frac{40n^3+84n^2}{4n^2+20n+21} + \frac{973n}{90}$
(vii) $F(CTW_n) = 8n^3 + 378n.$

Proof:

The M-polynomial of a Crown triangulated wheel graph CTW_n is given by
$M(CTW_n;x,y) = nx^7y^{2n} + nx^3y^{2n} + nx^7y^7 + 2nx^3y^7 + 2nx^2y^7.$
Using the expressions from Table 1, we have

$D_x = x\frac{\partial f(x,y)}{\partial x} = 7nx^7y^{2n} + 3nx^3y^{2n} + 7nx^7y^7 + 6nx^3y^7 + 4nx^2y^7$

$D_y = y\frac{\partial f(x,y)}{\partial y} = 2n^2x^7y^{2n} + 2n^2x^3y^{2n} + 7nx^7y^7 + 14nx^3y^7 + 14nx^2y^7$

$S_x = \int_0^x \frac{f(t,y)}{t}dt = \frac{nx^7y^{2n}}{7} + \frac{nx^3y^{2n}}{3} + \frac{nx^7y^7}{7} + \frac{2nx^3y^7}{3} + nx^2y^7$

$S_y = \int_0^y \frac{f(x,t)}{t}dt = \frac{x^7y^{2n}}{2} + \frac{x^3y^{2n}}{2} + \frac{nx^7y^7}{7} + \frac{2nx^3y^7}{7} + \frac{2nx^2y^7}{7}$

Therefore,

$M_1(CTW_n) = (D_x + D_y)(M(CTW_n;x,y))|_{x=y=1} = 4n^2 + 62n$

$M_2(CTW_n) = (D_xD_y)(M(CTW_n;x,y))|_{x=y=1} = 20n^2 + 119n$

$M_2^m(CTW_n) = (S_xS_y)(M(CTW_n;x,y))|_{x=y=1} = \frac{38n+35}{147}$

$SDD(CTW_n) = (D_xS_y + D_yS_x)(M(CTW_n;x,y))|_{x=y=1} = \frac{20n^2+317n+105}{21}$

$H(CTW_n) = 2S_xJ(M(CTW_n;x,y))|_{x=1} = \frac{8n^2+20n}{4n^2+20n+21} + \frac{311n}{315}$

$$ISI(CTW_n) = 2S_xJD_xD_y(M(CTW_n;x,y))|_{x=1} = \frac{40n^3+84n^2}{4n^2+20n+21} + \frac{973n}{90}$$
$$F(CTW_n) = (D_xD_x + D_yD_y)(M(CTW_n;x,y))|_{x=y=1} = 8n^3 + 378n.$$

IV. CONCLUSION:

In this paper we have obtained M-polynomials of some wheel related graphs using this some degree based topological indices of these graphs are derived. The advantage of M-polynomial is that, from that one expression we can obtain several degree-based topological indices.

REFERENCES:

[1] Aljohani, M.; Daoud, S.N. 2024. Edge Odd Graceful Labeling in Some Wheel-Related Graphs. Mathematics 12,1203: 1-31.
[2] Aljohani, M.; Daoud,S.N. 2025.The Edge Odd Graceful Labeling of Water Wheel Graphs. Axioms,14, 5.1-17
[3] E. Deutsch and S. Klavžar. 2015. M-Polynomial and degree-based topological indices, Iran. J. Math. Chem. 6 (2): 93−102.
[4] Fajtlowicz, S.1987.On conjectures of Graffitiff I. Congr. Numer. 60: 187–197
[5] E. J. Farrell. 1979. An introduction to matching polynomials, J. Combin. Theory Ser. B 27: 75−86.
[6] B. Furtula, I. Gutman. 2015. A forgotten topological index, Journal of Mathematical Chemistry, 53:1184-1190.
[7] I. Gutman, N. Trinajstic. 1972 .Graph theory and molecular orbitals. Total-electron energy of alternant hydrocarbons, Chem Phys. Lett. , 17: 535–538.
[8] F. Hassani, A. Iranmanesh and S. 2013. Mirzaie, Schultz and modified Schultz polynomials of fullerene, MATCH Commun. Math. Comput. Chem. 69: 87−92.
[9] H. Hosoya. 1988. On some counting polynomials in chemistry, Discrete Appl. Math. 19: 239−257.
[10] K.Pattabiraman. 2018. Inverse sum indeg index of graphs, International general of graphs and Combinatorics, 15:155-167.
[11] D. Vukiˇcevi´ c, M. Gaˇ sperov. 2010. Bond additive modeling 1. Adriatic indices, Croat. Chem. Acta 83: 243–260.