



## R#-Closed Sets in Digital Topology Toward Noise-Tolerant Region Analysis

Raghavendra K<sup>1</sup>, Veerasha A Sajjanara<sup>2</sup>, Govardhanareddy H G<sup>3</sup>

<sup>1</sup>Assistant professor, Department of Science and Humanities, S-VYASA Deemed to be University, Bangalore

<sup>2</sup>Associate Professor, Department of Mathematics, Presidency University, Bangalore

<sup>3</sup>Assistant Professor, Alliance School of Sciences, Alliance University, Bangalore

### Abstract

This paper examines R#-closed sets, a class of generalized closed sets in topology situated between g-closed and rg-closed sets, and proposes a framework for instantiating them within digital (Khalimsky) topological spaces used to model pixel grids and other discrete data domains. After reviewing the hierarchy of generalized closed sets and the foundations of digital topology, the chapter formally defines digital R#-closed sets, proves that the closure-under-union property of R#-closed sets carries over unchanged to the digital setting, and establishes a parity condition under which rectangular pixel blocks are guaranteed to be R#-closed. Building on these results, the chapter proposes R#-closed regions as a tunable middle ground between the strict requirements of digital g-closed sets and the permissive requirements of digital rg-closed sets, with potential relevance to block-based image partitioning, boundary-stability analysis, and discrete data topology more broadly. The chapter closes by identifying open problems for empirical and algorithmic follow-up work in computer science and engineering.

### Keywords

R#-closed sets; generalized closed sets; g-closed sets; rg-closed sets; digital topology; Khalimsky space; image processing; boundary-stability analysis; Alexandrov spaces; topological data analysis

### Classification

**2020 Mathematics Subject Classification (MSC):** 54A05 (topological spaces and generalizations, closure spaces); 54A10 (comparison of topologies); 54D10 (lower separation axioms); 54H30 (applications of general topology to computer science, e.g., digital topology, image processing).

**ACM Computing Classification System (CCS):** Mathematics of computing → Topology; Computing methodologies → Image processing; Theory of computation → Design and analysis of algorithms.

### Introduction

The study of generalized closed sets has been an active subfield of general topology since Levine (1970) introduced generalized closed (g-closed) sets, weakening the classical notion of a closed set by testing containment only against open supersets rather than requiring literal closedness. This relaxation proved fruitful: it produced a family of related notions, each calibrated against a different class of "test" open sets, including regular generalized closed (rg-closed) sets (Palaniappan & Rao, 1993) and R\*-closed sets (Janaki & Thomas, 2012). Ittanagi and Raghavendra (2017, 2018) extended this lineage by introducing R#-closed and R#-open sets, showing that the new class occupies a strict position between g-closed sets and rg-closed sets, and that it is closed under finite unions.

This lineage of definitions is, on its face, a purely internal development within general topology: each new class is motivated by filling a gap in a hierarchy of prior classes, and the resulting properties are proved for arbitrary topological spaces without reference to any particular application. The motivation for the present chapter is that this internal, general development becomes directly relevant to computer science and engineering once it is specialized to a topological space that is itself a model of discretized, real-world data — which is exactly what Khalimsky spaces provide for pixel grids. A property such as closure under finite unions, proved abstractly by Ittanagi and Raghavendra (2017) for arbitrary  $(X, \tau)$ , becomes a statement about combining image regions once  $(X, \tau)$  is taken to be a

finite Khalimsky subspace; and a strictness result placing  $R\#$ -closed sets strictly between  $g$ -closed and  $rg$ -closed sets becomes a statement about a tunable strictness dial for a region-acceptance criterion. Making this translation precise, rather than leaving it at the level of analogy, is the specific task this chapter undertakes.

Independently of this line of work, digital topology has developed methods for treating discrete grids — most importantly, pixel grids used in digital images — as topological spaces rather than merely as combinatorial or graph-theoretic structures. The Khalimsky line, and its higher-dimensional products, provide a rigorous topological substrate for digital images that supports notions such as digital connectivity, digital straight lines, and a genuine digital Jordan curve theorem (Khalimsky et al., 1990; Kong & Rosenfeld, 1989; Kovalevsky, 1989; Rosenfeld, 1979).

Despite the maturity of both research programs, the specific position of  $R\#$ -closed sets within digital topological spaces has not been worked out in detail. A recent working paper (Krishnappa, 2025) surveys broad, largely conceptual application areas for  $R\#$ -closed sets, including artificial intelligence, quantum computing, and blockchain networks, but does not develop a formal digital-topology instantiation or prove results specific to pixel-grid spaces. The gap this leaves is not merely terminological: without a digital-space specialization, it is not clear whether the defining property of  $R\#$ -closed sets — closure under finite unions — survives the passage from a general topological space to a finite Khalimsky subspace, nor is it clear how a practitioner would recognize a digital  $R\#$ -closed set without re-deriving the  $R^*$ -open/ $g$ -closed machinery from first principles for every candidate region. This chapter addresses that gap directly.

The chapter's contributions are fourfold. First, it formally defines digital  $R\#$ -closed sets within finite Khalimsky product spaces (Definition 2), fixing the notation used throughout the rest of the chapter and making explicit that the definition inherits, rather than modifies, the general topological one. Second, it verifies which properties of  $R\#$ -closed sets specialize correctly to the digital case, with complete proofs rather than assertions; in particular, Corollary 1 shows that closure under finite unions carries over to digital  $R\#$ -closed sets without additional hypotheses. Third, it identifies a concrete, coordinate-checkable family of digital  $R\#$ -closed sets arising from parity-aligned rectangular pixel blocks (Proposition 2), and shows, through worked examples, both how this family behaves under union (Worked Example 3) and how it fails for blocks whose boundaries are not parity-aligned (Worked Example 2). Fourth, it proposes, at a conceptual level, how  $R\#$ -closed regions might serve computer science and engineering (CSE) applications such as block-based image partitioning and boundary-stability analysis, while being explicit throughout about what remains to be empirically validated rather than presenting proposals as established results.

The remainder of the chapter is organized as follows. The Background section reviews the hierarchy of generalized closed sets, including a concrete illustration that the  $g$ -closed class is strictly larger than the closed class, and the fundamentals of digital topology. The Preliminaries section fixes notation and definitions and works through a small illustrative example. The Proposed Framework section develops digital  $R\#$ -closed sets, proves the parity criterion, and extends it to non-rectangular regions via unions. The Applications section discusses potential CSE relevance, including an algorithmic treatment of the parity test and its computational cost. The chapter closes with a discussion synthesizing the results, limitations, a research agenda, and concluding remarks.

## Background and Related Work

### Generalized Closed Sets

Let  $(X, \tau)$  be a topological space. A subset  $A$  of  $X$  is closed if  $X \setminus A$  is open, and the closure  $\text{cl}(A)$  is the smallest closed set containing  $A$ . Levine (1970) defined  $A$  to be  $g$ -closed if  $\text{cl}(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $X$ ; the complement of a  $g$ -closed set is  $g$ -open. Because every closed set is trivially  $g$ -closed ( $\text{cl}(A) = A$  whenever  $A$  is closed), while the converse fails in general, the  $g$ -closed sets form a strictly larger class than the closed sets. This relaxation is useful precisely because many topological arguments only require  $\text{cl}(A)$  to stay inside a prescribed open neighborhood of  $A$ , not that  $A$  itself be closed.

**Illustration.** The strictness of the containment is easy to see directly. Let  $X = \{a, b, c, d\}$  with  $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ , so that the closed sets are  $X, \{b, c, d\}, \{c, d\}$ , and  $\emptyset$ . The singleton  $\{c\}$  is not closed, since it does not appear in this list. It is, however,  $g$ -closed: the only open set containing  $\{c\}$  is  $X$  itself (none of  $\emptyset, \{a\}$ , or  $\{a, b\}$  contain  $c$ ), and  $\text{cl}(\{c\}) = \{c, d\} \subseteq X$  holds trivially. Hence  $\{c\}$  is  $g$ -closed without being closed, confirming that the  $g$ -closed class strictly contains the closed class in this space. This same four-point space is reused below (Example 1) to illustrate  $R\#$ -closedness directly, so that the digital examples that follow build on a single, fully worked-out finite model rather than a new space for every definition.

The hierarchy summarized in Table 1 below is only the part of a considerably larger family of generalized closed-set notions that is directly relevant to  $R\#$ -closed sets; it is not the whole family studied in this literature. Using semi-open sets (Levine, 1963) as the test class in place of open sets, Bhattacharyya and Lahiri (1987) introduced semi-generalized closed (sg-closed) sets, and Arya and Nour (1990) introduced the complementary generalized semi-closed (gs-closed) sets, both of which sit alongside, rather than strictly inside, the  $g$ -closed/ $rg$ -closed/ $R\#$ -closed chain this chapter focuses on. These parallel branches of the hierarchy are not developed further here, since doing so would require a separate digital-space specialization argument for each; they are noted only to make clear that the four-class hierarchy in Table 1 is a deliberately chosen slice of a richer landscape, selected because it is the slice in which  $R\#$ -closed sets sit.

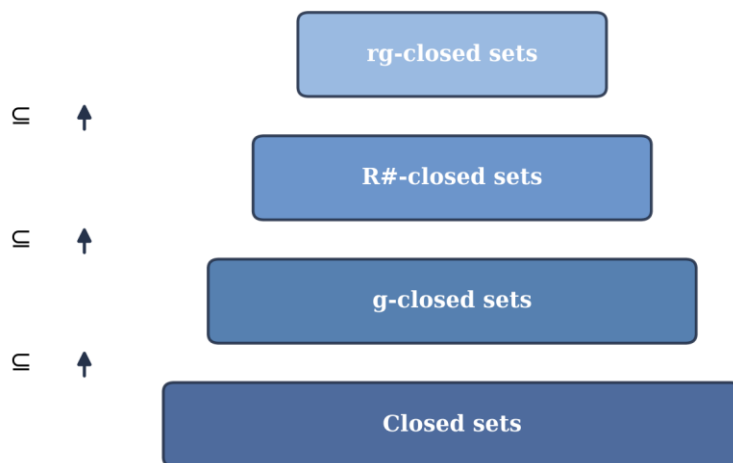
Subsequent work generalized this idea further by widening the class of "test" open sets against which containment is checked. A set  $U$  is regular open if  $U = \text{int}(\text{cl}(U))$ ;  $A$  is  $rg$ -closed (Palaniappan & Rao, 1993) if  $\text{cl}(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is regular open. Because every open set need not be regular open, but every regular open set is open, testing against regular open sets only is a weaker requirement

on  $A$ , so the class of  $rg$ -closed sets is strictly larger than the class of  $g$ -closed sets. Janaki and Thomas (2012) introduced  $R^*$ -closed sets using the regular closure operator  $rcl$  (the smallest regular closed set containing a given set) tested against  $rg$ -open sets.

Ittanagi and Raghavendra (2017) combined these ingredients to define  $R\#$ -closed sets: a subset  $A$  of  $(X, \tau)$  is  $R\#$ -closed if  $gcl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $R^*$ -open, where  $gcl(A)$  denotes the generalized closure of  $A$  — the intersection of all  $g$ -closed sets containing  $A$ . The companion notion of  $R\#$ -open sets was developed in a follow-up study (Ittanagi & Raghavendra, 2018). The authors established that every  $g$ -closed set is  $R\#$ -closed, that every  $R\#$ -closed set is  $rg$ -closed, and that  $R\#$ -closed sets are closed under finite unions, placing the new class strictly between  $g$ -closed and  $rg$ -closed sets in the standard hierarchy of weak closed-set notions.

It is this last property — closure under finite unions — that makes  $R\#$ -closed sets an attractive candidate for the region-based applications this chapter is concerned with, more so than either endpoint of the hierarchy on its own. Closedness itself is well behaved under union (a union of finitely many closed sets is closed), but is too restrictive a starting condition for noisy, discretized data, as the Illustration above and the digital examples later in this chapter make concrete.  $rg$ -closedness is permissive enough to accommodate more candidate regions, but a hierarchy positioned only at that permissive end offers no guarantee against accepting regions whose boundaries are unstable under small perturbations.  $R\#$ -closedness inherits closure under union from its position in the hierarchy (Proposition 1) while remaining strictly closer to the  $g$ -closed, more restrictive end than  $rg$ -closed sets are in general — which is exactly the combination of properties a region-combination criterion for image analysis needs: regions can be built up piecewise by union (as in Worked Example 3 below) without losing the strictness that boundary-stability analysis is intended to enforce.

### Hierarchy of Generalized Closed Sets



Each class strictly contains the one below it (Levine, 1970; Ittanagi & Raghavendra, 2017; Palaniappan & Rao, 1993).  $R\#$ -closed sets sit strictly between  $g$ -closed and  $rg$ -closed sets.

**Figure 1.** The nested hierarchy of generalized closed sets relevant to this chapter. Moving outward from closed sets relaxes the containment requirement step by step;  $R\#$ -closed sets are the second-widest class shown, strictly containing all  $g$ -closed sets and strictly contained in all  $rg$ -closed sets.

### Comparing the Generalized Closed-Set Classes

Table 1 summarizes the four classes reviewed above along the dimensions most relevant to the rest of this chapter: the test open sets used in each definition, the closure operator each definition checks, and the strictness of containment relative to the ordinary closed sets. The rightmost column previews the interpretation developed in the Applications section, where the position of a class within this hierarchy is read as a proxy for how strict, and therefore how noise-tolerant, a corresponding image-region criterion would be.

**Table 1.** The hierarchy of generalized closed-set classes reviewed in this chapter, ordered from strictest (closed) to most permissive ( $rg$ -closed).

| Class         | Test open sets    | Closure operator checked | Relative strictness                           |
|---------------|-------------------|--------------------------|-----------------------------------------------|
| Closed        | — (definitional)  | $cl(A) = A$              | Strictest                                     |
| $g$ -closed   | Open sets         | $cl(A)$                  | Strictly weaker than closed                   |
| $R\#$ -closed | $R^*$ -open sets  | $gcl(A)$                 | Strictly between $g$ -closed and $rg$ -closed |
| $rg$ -closed  | Regular open sets | $cl(A)$                  | Most permissive of the four                   |

Two features of Table 1 are worth making explicit. First,  $R\#$ -closedness is not simply "closedness tested against a smaller set of open sets" — unlike  $g$ -closed and  $rg$ -closed, which both check the ordinary closure  $cl(A)$ ,  $R\#$ -closed checks the generalized closure  $gcl(A)$ , which is itself defined in terms of the  $g$ -closed class. This nesting of definitions is why Ittanagi and Raghavendra's (2017) placement of  $R\#$ -closed strictly between  $g$ -closed and  $rg$ -closed is a nontrivial result rather than an immediate consequence of restricting the test sets. Second, because the table is organized by strictness rather than by chronology, it also functions as a map of the proof obligations in the Preliminaries section below: establishing that  $R\#$ -closed sets are closed under union (Proposition 1) requires reasoning about  $gcl$ , not  $cl$ , which is why the proof reproduced there tracks  $gcl(A) \cup gcl(B)$  explicitly rather than appealing to the corresponding fact for  $cl$ .

### Digital Topology and the Khalimsky Space

Digital topology asks how classical topological notions — connectedness, boundary, continuity — can be made precise on discrete grids such as pixel arrays, rather than only on continuous spaces. Rosenfeld (1979) and Kong and Rosenfeld (1989) developed a graph-theoretic approach based on adjacency relations, while Khalimsky and collaborators developed a genuinely topological approach (Khalimsky et al., 1990; Kovalevsky, 1989). For finite digital spaces, the two approaches are known to be equivalent. The topological approach is the one this chapter builds on, precisely because the generalized-closed-set hierarchy summarized in Table 1 is defined in terms of open and closed sets; a purely graph-theoretic adjacency model would first need to be translated into a topology (for example, via the Alexandrov-topology construction discussed later in this chapter) before any of Definition 1's four classes could be applied to it at all.

The Khalimsky line equips the integers  $\mathbb{Z}$  with the topology in which a set is open exactly when, for every even integer it contains, it also contains that integer's two neighbors. Equivalently, the minimal open neighborhood of an odd integer  $n$  is the singleton  $\{n\}$ , while the minimal open neighborhood of an even integer  $n$  is the triple  $\{n-1, n, n+1\}$ . Odd points are therefore open points and even points are closed points (the singleton  $\{n\}$  is closed exactly when  $n$  is even). The Khalimsky plane is the product topology on  $\mathbb{Z} \times \mathbb{Z}$  obtained from two copies of the Khalimsky line, and higher-dimensional Khalimsky spaces are defined analogously. A digital image can then be modeled as a finite subspace of the Khalimsky plane, with pixel coordinates as points and the induced subspace topology encoding adjacency. This construction is what allows classical topological machinery — including generalized-closed-set notions — to be applied directly to image regions rather than merely by analogy.

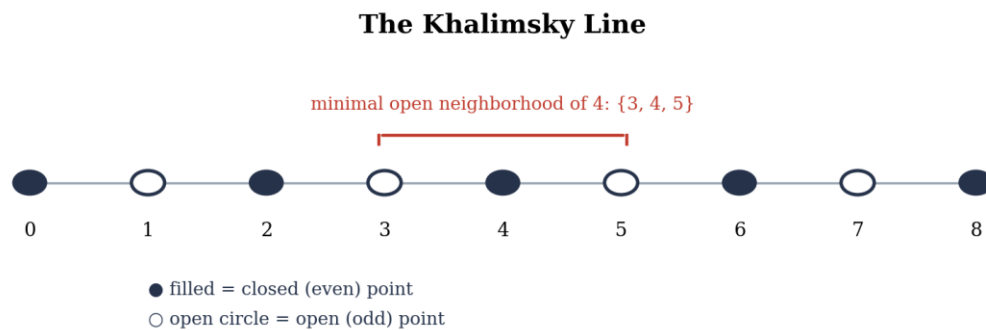


**Figure 2.** The Khalimsky line on the integers 0 through 8. Filled points sit at even coordinates and are closed points; open circles sit at odd coordinates and are open points. Every even point's minimal open neighborhood is the triple of itself and its two neighbors, illustrated here for the point 4.

### Preliminaries

This section fixes the definitions used throughout the chapter and works through a small finite example, following the convention in this literature of illustrating new set classes on small spaces before moving to larger or infinite ones. Definition 1 collects the four closed-set notions introduced narratively in the Background section into a single, uniform statement, so that later sections can refer back to a single numbered source rather than to scattered prose definitions; this also makes explicit the nesting structure noted in Table 1, where each successive part of Definition 1 tests containment against a wider class of open sets, or a different closure operator, than the part before it.

**Definition 1.** Let  $(X, \tau)$  be a topological space and  $A \subseteq X$ . Then:

- (i)  $A$  is  $g$ -closed if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open.
- (ii)  $U$  is regular open if  $U = \text{int}(cl(U))$ ;  $A$  is  $rg$ -closed if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is regular open.
- (iii)  $A$  is  $R^*$ -closed if  $rcl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $rg$ -open (the complement of an  $rg$ -closed set), where  $rcl(A)$  is the smallest regular closed set containing  $A$ .
- (iv)  $A$  is  $R\#$ -closed if  $gcl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $R^*$ -open (the complement of an  $R^*$ -closed set), where  $gcl(A)$  is the intersection of all  $g$ -closed sets containing  $A$ .

**Proposition 1 (Ittanagi & Raghavendra, 2017).** If  $A$  and  $B$  are  $R\#$ -closed subsets of a topological space  $(X, \tau)$ , then  $A \cup B$  is  $R\#$ -closed.

This chapter reproduces the proof in full because the digital-space specialization argument in a later section depends on it.

**Proof.** First note that finite unions of g-closed sets are g-closed: if  $A$  and  $B$  are g-closed and  $U$  is open with  $A \cup B \subseteq U$ , then  $A \subseteq U$  and  $B \subseteq U$ , so  $\text{cl}(A) \subseteq U$  and  $\text{cl}(B) \subseteq U$  by g-closedness of each; since  $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$  for any two subsets of a topological space,  $\text{cl}(A \cup B) \subseteq U$ , so  $A \cup B$  is g-closed. Consequently  $\text{gcl}(A) \cup \text{gcl}(B)$  is g-closed, and since it contains  $A \cup B$ , minimality of  $\text{gcl}(A \cup B)$  as the smallest g-closed superset of  $A \cup B$  gives  $\text{gcl}(A \cup B) \subseteq \text{gcl}(A) \cup \text{gcl}(B)$ . Now let  $U$  be  $R^*$ -open with  $A \cup B \subseteq U$ . Then  $A \subseteq U$  and  $B \subseteq U$ , so  $R\#$ -closedness of  $A$  and  $B$  gives  $\text{gcl}(A) \subseteq U$  and  $\text{gcl}(B) \subseteq U$ , hence  $\text{gcl}(A) \cup \text{gcl}(B) \subseteq U$ , and therefore  $\text{gcl}(A \cup B) \subseteq U$ . Thus  $A \cup B$  is  $R\#$ -closed. ■

**Example 1.** Let  $X = \{a, b, c, d\}$  with  $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ . The closed sets are  $X, \{b, c, d\}, \{c, d\}$ , and  $\emptyset$ . Direct computation shows every closed set here is g-closed (as is always the case), and one can verify that  $\{c, d\}$  is  $R\#$ -closed: the only  $R^*$ -open sets containing  $\{c, d\}$  are  $\{b, c, d\}$  and  $X$ , and  $\text{gcl}(\{c, d\}) = \{c, d\}$  (since  $\{c, d\}$  is already closed, hence g-closed), which is contained in both. This small space illustrates the mechanics that the digital examples in a later section reuse at a larger scale.

### *The Khalimsky Space as a Digital Topological Framework*

To connect the preceding machinery to image data, this section records one further known fact about the Khalimsky line that is used repeatedly below: a finite interval  $[a, b] = \{a, a + 1, \dots, b\}$  of the Khalimsky line is closed if and only if both endpoints  $a$  and  $b$  are even integers.

This follows directly from the definition of the topology. The complement  $Z \setminus [a, b]$  is a union of two rays. For this complement to be open, every even integer it contains must have both neighbors in the complement. The only integers at risk of violating this are the boundary points  $a - 1$  and  $b + 1$ . If  $a - 1$  is even (equivalently,  $a$  is odd), its right neighbor  $a$  lies in  $[a, b]$  and not in the complement, so the complement fails to be open; the same argument applies at  $b + 1$  when  $b$  is odd. Conversely, when  $a$  and  $b$  are both even,  $a - 1$  and  $b + 1$  are odd and impose no adjacency requirement, and every other even integer in the complement already has both of its neighbors in the same ray. Hence  $[a, b]$  is closed exactly when  $a$  and  $b$  are even. This is consistent with the standard description of Khalimsky arcs and closed digital intervals in the digital-topology literature (Khalimsky et al., 1990; Kovalevsky, 1989).

The Khalimsky plane is the product topology of two Khalimsky lines, and a digital image domain is modeled as a finite rectangular subspace of it, with each pixel a point of the space and adjacency encoded by the topology rather than assumed separately. The next section uses the parity fact above, together with the standard product-topology identity  $\text{cl}(A \times B) = \text{cl}(A) \times \text{cl}(B)$ , to identify a concrete family of digital  $R\#$ -closed sets.

It is worth pausing on why this parity fact is the right lever for the rest of the chapter, rather than an arbitrary computational shortcut. In a general topological space, deciding whether a set is closed requires knowing the full topology — the entire collection of open sets — since closedness is defined as the complement of membership in that collection. In a Khalimsky subspace, by contrast, the topology itself is generated by a single, easily stated coordinate-parity rule, so closedness of an interval reduces to inspecting two integers rather than searching a collection of sets. This is a special feature of the Khalimsky construction, not a property shared by digital topologies in general (a graph-theoretic adjacency model, for instance, does not come with a comparably simple coordinate test). The framework developed in the next section depends on this feature: Proposition 2 below is only as computationally cheap as it is because the underlying parity fact is.

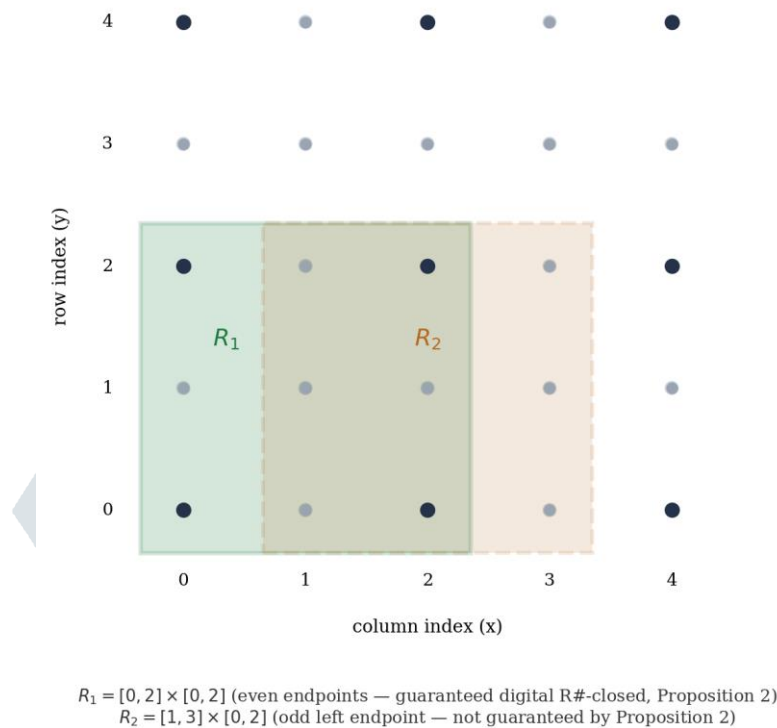
### **Proposed Framework: Digital $R\#$ -Closed Sets**

**Definition 2 (Digital  $R\#$ -closed set).** Let  $D$  be a finite subspace of the Khalimsky plane ( $Z \times Z, \kappa \times \kappa$ ), representing a digital image domain, with the subspace topology. A subset  $A$  of  $D$  is a digital  $R\#$ -closed set if it is  $R\#$ -closed in  $D$  in the sense of Definition 1(iv); that is,  $\text{gcl}(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $R^*$ -open in  $D$ .

Because Definition 2 does not add any hypothesis beyond  $D$  being a topological space, Proposition 1 applies without modification: the union of any two digital  $R\#$ -closed sets in  $D$  is itself a digital  $R\#$ -closed set. This is recorded as Corollary 1 for completeness.

**Corollary 1.** If  $A$  and  $B$  are digital  $R\#$ -closed sets in a Khalimsky subspace  $D$ , then  $A \cup B$  is a digital  $R\#$ -closed set in  $D$ .

### A 5×5 Khalimsky Pixel Grid



The more substantive question for applications is how to recognize digital  $R\#$ -closed sets concretely, without recomputing the full  $R^*$ -open/g-closed machinery for every candidate region. Proposition 2 identifies a simple, checkable sufficient condition based only on pixel coordinates.

**Proposition 2 (Parity-aligned rectangles are digital  $R\#$ -closed).** Let  $D$  be a rectangular subspace of the Khalimsky plane, and let  $R = [a, b] \times [c, d] \subseteq D$  be a rectangular block of pixel coordinates. If  $a, b, c$ , and  $d$  are all even integers, then  $R$  is a digital  $R\#$ -closed set.

**Proof.** By the parity fact established in the previous section,  $[a, b]$  is closed in the Khalimsky line when  $a$  and  $b$  are even, and likewise  $[c, d]$  is closed when  $c$  and  $d$  are even. Since closure in a finite product topology satisfies  $\text{cl}(A \times B) = \text{cl}(A) \times \text{cl}(B)$ , the rectangle  $[a, b] \times [c, d]$  is closed in the product (Khalimsky plane) topology, and this remains true in the subspace  $D$ . Every closed set is g-closed (taking  $\text{cl}(A) = A$  in Definition 1(i) satisfies the defining inequality trivially for any open superset). By the hierarchy established by Ittanagi and Raghavendra (2017), every g-closed set is  $R\#$ -closed. Hence  $R$  is a digital  $R\#$ -closed set. ■

**Worked Example 2.** Let  $D = \{0, 1, 2, 3, 4\} \times \{0, 1, 2, 3, 4\}$  represent a  $5 \times 5$  pixel block. The rectangle  $R_1 = [0, 2] \times [0, 2]$  has all-even endpoints (0 and 2), so Proposition 2 guarantees  $R_1$  is a digital  $R\#$ -closed set regardless of the specific pixel values inside it. By contrast,  $R_2 = [1, 3] \times [0, 2]$  has an odd endpoint ( $a = 1$ ), so Proposition 2 does not apply to  $R_2$ ; whether  $R_2$  is  $R\#$ -closed would have to be checked directly against Definition 1, and in general it need not be. This asymmetry is a direct, checkable consequence of how the pixel grid is anchored to the Khalimsky coordinate parity, not an artifact of pixel content.

**Figure 3.** A  $5 \times 5$  Khalimsky pixel grid illustrating Worked Example 2. Darker points mark even-even (closed) coordinates. Region  $R_1$  (green, solid) has all-even boundary coordinates and is therefore guaranteed digital  $R\#$ -closed by Proposition 2. Region  $R_2$  (orange, dashed) has an odd left boundary and receives no such guarantee, even though it overlaps  $R_1$  and is the same size and shape.

**Worked Example 3 (non-rectangular regions via Corollary 1).** Proposition 2 only guarantees  $R\#$ -closedness for individual rectangles, but Corollary 1 extends this guarantee to unions of such rectangles, which need not themselves be rectangular. Reusing  $D = \{0, 1, 2, 3, 4\} \times \{0, 1, 2, 3, 4\}$  from Worked Example 2, let  $R_1 = [0, 2] \times [0, 2]$  as before, and let  $R_3 = [2, 4] \times [2, 4]$ . Both  $R_1$  and  $R_3$  have all-even boundary coordinates, so Proposition 2 guarantees each is individually a digital  $R\#$ -closed set. The region  $S = R_1 \cup R_3$  — two  $2 \times 2$  blocks touching only at the corner pixel (2, 2) — is not itself a rectangle, so Proposition 2 does not apply to  $S$  directly. Corollary 1, however,

guarantees that  $S$  is a digital  $R\#$ -closed set regardless, since it is the union of two digital  $R\#$ -closed sets. This illustrates the practical value of Corollary 1: a library of  $R\#$ -closed "building block" rectangles, generated cheaply via Proposition 2, can be combined by union into  $R\#$ -closed regions of essentially arbitrary shape, without re-verifying Definition 1(iv) for each combined shape from scratch.

Two observations follow from Proposition 2 and Worked Example 3 that are relevant to the Applications section below. First, Proposition 2 gives a way to guarantee  $R\#$ -closedness by construction — by aligning block boundaries to even coordinates — rather than by exhaustive verification, which matters for any downstream use where regions need to be generated algorithmically rather than checked one at a time. Second, because  $R\#$ -closed sets sit strictly between  $g$ -closed and  $rg$ -closed sets, and closed sets are a still-narrower special case used in Proposition 2, the parity-aligned rectangles identified here (and their unions, per Corollary 1) are a conservative ( $g$ -closed, hence  $R\#$ -closed) family; a full characterization of the strictly  $R\#$ -closed-but-not- $g$ -closed digital sets in a given domain  $D$  is left as an open problem in the Limitations and Future Directions section.

## Potential Applications in Computer Science and Engineering

The results above are theoretical; this section proposes, at a conceptual level, how they might inform CSE work, while being explicit that empirical validation is future work rather than a claim established here.

### Block-Based Image Partitioning

Many image-processing pipelines already partition images into fixed-size rectangular blocks (for compression, tiling, or parallel processing). Proposition 2 shows that when block boundaries are aligned to even Khalimsky coordinates — a design choice, not a constraint imposed by pixel content — every resulting block is guaranteed to be a digital  $R\#$ -closed set. This offers a topologically grounded justification, rather than a purely combinatorial one, for even-aligned block grids, and suggests that block-boundary placement could be chosen deliberately with this guarantee in mind in applications where the  $R\#$ -closed property itself is useful downstream (for instance, wherever a later stage of a pipeline is proved to behave predictably on  $R\#$ -closed inputs).

A practical consequence of Algorithm 1 is that this guarantee can be checked, or enforced, at negligible cost within an existing partitioning pipeline. A system that already tiles an image into blocks can apply the parity test to each block boundary as a cheap post-processing pass, flagging any block whose boundary is not parity-aligned for separate handling (for example, by nudging the boundary to the nearest even coordinate, at the cost of a one-pixel shift, or by falling back to a direct check against Definition 1). Because the test is  $O(1)$  per block, this pass adds negligible overhead relative to the block partitioning itself, which is at minimum linear in the number of pixels. Whether enforcing parity alignment in this way measurably improves downstream behavior — as opposed to merely being cheap to check — is, again, an empirical question left to future work.

### A Tunable Middle Ground for Boundary Stability

Because  $R\#$ -closed sets sit strictly between  $g$ -closed and  $rg$ -closed sets, they occupy a middle position between the strictest and most permissive members of this family. Requiring a candidate image region to be digital  $g$ -closed may be too strict when region boundaries are affected by sensor noise, while requiring only digital  $rg$ -closed may be too permissive, admitting regions whose boundaries are unstable under small perturbations. The  $R\#$ -closed condition, sitting strictly in between, is proposed here as a candidate criterion for boundary-stability analysis: a region satisfying it is guaranteed to satisfy the weaker  $rg$ -closed condition while remaining closer to the  $g$ -closed end of the hierarchy than  $rg$ -closed sets in general. Whether this intermediate position translates into a measurable robustness advantage on real image data — for example, against the classical morphological opening and closing operators developed within mathematical morphology (Serra, 1982) — is an empirical question outside the scope of this chapter and is proposed as a direction for future work.

To make the intended reading concrete without overstating it as an empirical claim, consider a hypothetical block-based segmentation pipeline that must decide, for each candidate block boundary, whether to treat it as final or to re-examine it under a noisier re-scan of the sensor. A pipeline using only the closed-set criterion would reject any block whose boundary drifts by even a single pixel between scans, discarding many blocks unnecessarily. A pipeline using only the  $rg$ -closed criterion would accept a wide range of boundary perturbations, including some that a human annotator would judge as genuinely unstable. A pipeline built on the digital  $R\#$ -closed criterion developed here — for instance, by requiring block boundaries to remain parity-aligned rectangles, or unions thereof, under Proposition 2 and Corollary 1 — would occupy the middle ground the chapter proposes: tolerant of the kind of boundary perturbation that leaves parity alignment intact, while still excluding the most permissive  $rg$ -closed regions. This scenario is offered purely to illustrate the intended use case; the chapter does not claim to have measured this trade-off on real sensor data, and doing so is identified as future work below.

### Beyond Pixel Grids: Discrete Data Topology

The Khalimsky construction is one instance of a more general fact: any finite discrete structure (a grid, a graph, a point cloud with a chosen adjacency or similarity relation) can be equipped with an Alexandrov topology — a topology in which arbitrary intersections of open sets remain open, first studied by Alexandroff (1937) — via a specialization preorder, after which the entire generalized-closed-set hierarchy —  $g$ -closed,  $rg$ -closed,  $R^*$ -closed,  $R\#$ -closed — applies without modification. This suggests that the framework developed here for pixel grids could, in principle, extend to other finite data structures used in CSE, such as graph-based representations in network analysis or the simplicial complexes used in topological data analysis, where persistent homology (Zomorodian & Carlsson, 2005) is

already used to extract multi-scale topological features from discrete point-cloud data. Working out this extension rigorously for a specific data structure, rather than pixel grids, is left for future work.

What would be needed to carry out such an extension is worth being explicit about, since it clarifies why this chapter restricts itself to the Khalimsky case rather than working at the level of generality of Alexandrov spaces throughout. Definition 2 and Proposition 2 both rely on two specific facts about the Khalimsky plane: that it is a product of two copies of a single, well-understood one-dimensional space (the Khalimsky line), and that closedness in that one-dimensional space reduces to a simple parity test on interval endpoints. A graph equipped with an Alexandrov topology via a specialization preorder need not decompose as a product of simpler spaces, and need not admit any comparably simple test for closedness of a candidate vertex subset. Extending Proposition 2's style of result — a cheap, coordinate-level sufficient condition for  $R\#$ -closedness — to a graph or point-cloud setting would therefore likely require identifying an analogous structural feature specific to the chosen discrete structure, rather than a direct restatement of the rectangle argument used here.

### Computational Considerations

Because Khalimsky subspaces used to model images are finite, every point has a unique minimal open neighborhood, and membership in the various closed-set classes reduces to checking a bounded number of candidate neighborhoods rather than searching over all open sets. This suggests that testing whether a given region is a digital  $R\#$ -closed set is computationally tractable for image-sized domains, though a precise complexity analysis and reference implementation have not been carried out here and are identified as a concrete next step below. It is useful to separate two distinct computational questions: testing the sufficient condition of Proposition 2 against a candidate rectangle, and testing full  $R\#$ -closedness against Definition 1(iv) for an arbitrary candidate region.

**Algorithm 1 (parity-aligned rectangle test).** Given a candidate rectangle  $R = [a, b] \times [c, d]$  within a Khalimsky subspace  $D$ , the test implied by Proposition 2 is:

1. Read the four boundary coordinates  $a, b, c, d$ .
2. Check whether  $a \bmod 2 = 0, b \bmod 2 = 0, c \bmod 2 = 0$ , and  $d \bmod 2 = 0$ .
3. If all four checks pass, report  $R$  as digital  $R\#$ -closed by Proposition 2.
4. Otherwise, report that Proposition 2 does not apply to  $R$  ( $R\#$ -closedness would need to be checked directly against Definition 1, and in general need not hold, as Worked Example 2 shows for  $R_2$ ).

Because Algorithm 1 only inspects four integers regardless of how many pixels  $R$  contains, its cost is  $O(1)$  per candidate rectangle, and testing every rectangle in a fixed block-partitioning scheme costs time linear in the number of blocks. This is the sense in which Proposition 2 is computationally attractive: it replaces a topological verification with an arithmetic parity check.

The second question — testing an arbitrary candidate region  $A \subseteq D$  for  $R\#$ -closedness directly against Definition 1(iv), without assuming it is a parity-aligned rectangle or a union of such rectangles — is considerably more expensive in the worst case. A direct implementation would need to enumerate the  $R^*$ -open supersets  $U$  of  $A$  within  $D$  and confirm  $\text{gcl}(A) \subseteq U$  for each; since a finite space on  $n$  points can have as many as  $2^n$  distinct subsets, and in the worst case a correspondingly large number of open sets, a naive enumeration is exponential in the number of pixels in  $D$ . Whether a polynomial-time algorithm exists for the general test — for instance, by exploiting the specific combinatorial structure of the Khalimsky topology rather than treating  $D$  as an arbitrary finite space — is an open question that this chapter does not resolve, and it is listed explicitly as a research-agenda item below.

### Limitations and Future Directions

This chapter is deliberately theoretical in scope: it establishes definitions and proves properties of digital  $R\#$ -closed sets but does not report experiments on image datasets, nor does it benchmark the proposed criterion against established segmentation or morphological methods. Several directions follow naturally from the results above and are proposed as a research agenda rather than claims of the chapter itself:

Characterizing digital  $R\#$ -closed sets that are not already  $g$ -closed. Proposition 2 identifies only a sufficient, not a necessary, condition: it guarantees  $R\#$ -closedness for parity-aligned rectangles (and, via Corollary 1, their unions), but says nothing about whether there exist digital  $R\#$ -closed sets that are neither closed, nor  $g$ -closed, nor a union of parity-aligned rectangles. Determining how much larger the full digital  $R\#$ -closed family is than the constructive family identified here would clarify whether Algorithm 1 captures most of the practically useful cases or only a narrow slice of them.

Developing and empirically evaluating an algorithm that tests or constructs digital  $R\#$ -closed regions on real image data. The worst-case exponential cost identified for the general Definition 1(iv) test in the Computational Considerations section makes clear that a naive implementation will not scale to image-sized domains; a concrete next step is to design an algorithm that either restricts attention to a tractable sub-family (as Proposition 2 and Corollary 1 already do) or exploits the specific combinatorial structure of the Khalimsky topology to test the general case more efficiently, followed by a complexity analysis and a comparison of its behavior under synthetic and real sensor noise against the classical morphological opening and closing operators of Serra (1982).

Extending the definitions to  $R\#$ -continuous maps between digital spaces. The chapter treats  $R\#$ -closedness only as a static property of a fixed region within a fixed domain  $D$ ; relating digital  $R\#$ -closed sets across different image resolutions, or under geometric transformations such as rotation and scaling, would require a compatible notion of continuity between Khalimsky spaces, which has not been developed here and is a prerequisite for any multi-resolution application of the framework.

Working out the specialization of the framework to a non-grid discrete structure, such as a graph or point cloud equipped with an Alexandrov topology via a specialization preorder. Section 6 asserts, at a conceptual level, that the generalized-closed-set hierarchy applies without modification to any such structure once the Alexandrov topology is fixed, but this chapter carries out the detailed argument only for the Khalimsky grid case; confirming the claim rigorously for a specific non-grid structure, and identifying whether an analogue of Proposition 2 exists for that structure, is left as a direct extension of this work.

## Discussion

Taken together, the results developed in this chapter support a narrower and a broader claim, and it is worth keeping the two separate. The narrower claim is fully established by proof: digital  $R\#$ -closed sets are well defined within finite Khalimsky subspaces (Definition 2), the closure-under-union property proved for the general topological setting by Ittanagi and Raghavendra (2017) carries over to the digital setting without modification (Corollary 1), and parity-aligned rectangular blocks — along with unions of such blocks — form a concrete, coordinate-checkable family of digital  $R\#$ -closed sets (Proposition 2, Worked Examples 2 and 3). These results hold unconditionally within the stated framework and do not depend on any assumption about image content, noise model, or downstream algorithm.

The broader claim — that digital  $R\#$ -closed regions offer a useful, noise-tolerant middle ground for practical tasks such as block-based image partitioning and boundary-stability analysis — is proposed at a conceptual level and is explicitly not established here. Table 1 and the hypothetical scenario in the Applications section are intended to make the proposal precise enough to be tested, not to substitute for the test itself. The gap between the narrow and broad claims is exactly the gap identified in the Limitations and Future Directions section: what is missing is not a mathematical development of the  $R\#$ -closed hierarchy, which this chapter provides for the digital case, but an empirical and algorithmic demonstration that the hierarchy's intermediate position translates into a measurable advantage over existing digital  $g$ -closed or  $rg$ -closed criteria, or over non-topological baselines such as classical morphological operators, on real image data.

This distinction also clarifies how the chapter relates to Krishnappa (2025), the working paper that motivated this study. That paper surveys conceptual application areas for  $R\#$ -closed sets broadly, without a digital-topology instantiation; this chapter narrows the scope substantially — to a single, well-understood digital-topology substrate (the Khalimsky plane) — in exchange for being able to state and prove specific results (Definition 2, Corollary 1, Proposition 2) rather than conceptual possibilities. The trade-off is deliberate: a narrower, fully proved digital instantiation is intended to give later empirical work a precise, falsifiable starting point, rather than requiring that work to first formalize the digital setting from scratch.

## Conclusion

This chapter formally situated  $R\#$ -closed sets — a class of generalized closed sets introduced by Ittanagi and Raghavendra (2017, 2018) that lies strictly between  $g$ -closed and  $rg$ -closed sets — within the digital topological framework built on the Khalimsky line and plane. It showed that the closure-under-union property of  $R\#$ -closed sets specializes without modification to digital spaces (Corollary 1), and it identified a concrete, verifiable family of digital  $R\#$ -closed sets arising from rectangular pixel blocks whose boundaries are aligned to even Khalimsky coordinates (Proposition 2), together with a cheap algorithmic test for recognizing that family (Algorithm 1) and a demonstration that the family extends to non-rectangular regions by union (Worked Example 3). Building on these results, the chapter proposed — without claiming empirical validation — that digital  $R\#$ -closed regions may offer a useful, topologically principled middle ground between overly strict and overly permissive closure criteria for boundary-stability analysis in image processing, and that the same construction may generalize to other finite data structures used in computer science and engineering.

The Discussion section above deliberately separated what this chapter establishes by proof from what it proposes as a direction for further work, and that separation is the chapter's central methodological point as much as any individual result: a digital-topology instantiation of a generalized closed-set class can be made fully rigorous, computationally concrete, and still explicit about the limits of what has been shown. The open problems identified in the Limitations and Future Directions section — characterizing the full digital  $R\#$ -closed family, developing a tractable general-purpose test, extending the framework to  $R\#$ -continuous maps, and working out a non-grid instantiation — are offered as a starting point for the empirical and algorithmic work needed to test the proposals made here, and as concrete entry points for readers in computer science, engineering, or general topology who wish to build on this chapter's results directly.

## Key Terms and Definitions

**Closed Set:** In a topological space, a set whose complement is open; equivalently, a set that equals its own closure.

**$g$ -Closed Set:** A set  $A$  for which the closure of  $A$  is contained in every open set that contains  $A$ ; a weaker requirement than being closed.

**$rg$ -Closed Set:** A set  $A$  for which the closure of  $A$  is contained in every regular open set (a set equal to the interior of its own closure) that contains  $A$ .

**R\*-Closed Set:** A set  $A$  for which the regular closure of  $A$  is contained in every  $rg$ -open set that contains  $A$ .

**R#-Closed Set:** A set  $A$  for which the generalized closure of  $A$  is contained in every  $R^*$ -open set that contains  $A$ ; a class of sets lying strictly between  $g$ -closed and  $rg$ -closed sets.

**Khalimsky Topology:** A topology on the integers in which odd points are open and even points are closed, used as the standard topological model of a digital line, and extended to higher dimensions to model pixel grids.

**Digital R#-Closed Set:** An  $R^*$ -closed set defined within a finite subspace of the Khalimsky plane, used in this chapter to model a candidate region within a digital image.

**Alexandrov Space:** A topological space in which every point has a unique minimal open neighborhood; finite spaces, including Khalimsky subspaces, are always Alexandrov spaces.

**Parity-Aligned Rectangle:** A rectangular block of pixel coordinates whose boundary coordinates are all even integers in the Khalimsky plane, guaranteeing (by Proposition 2 of this chapter) that the block is closed, and therefore digital  $R^*$ -closed.

## References

- [1]. Alexandroff, P. S. (1937). Diskrete Räume. *Matematicheskii Sbornik*, 2(3), 501–518.
- [2]. Arya, S. P., & Nour, T. (1990). Characterizations of  $s$ -normal spaces. *Indian Journal of Pure and Applied Mathematics*, 21(8), 717–719.
- [3]. Bhattacharyya, P., & Lahiri, B. K. (1987). Semi-generalized closed sets in topology. *Indian Journal of Mathematics*, 29(3), 375–382.
- [4]. Ittanagi, B. M., & Raghavendra, K. (2017). On  $R^*$  closed sets in topological spaces. *Computer and Mathematical Sciences*, 8(11), 614–620.
- [5]. Ittanagi, B. M., & Raghavendra, K. (2018). On  $R^*$ -open sets in topological spaces. *International Journal of Engineering & Technology* 7 (4.10) (2018) 770-772
- [6]. Janaki, C., & Renu Thomas, A. (2012). On  $R^*$  closed sets in topological spaces. *International Journal of Mathematical Archive*, 3(8), 3067–3074.
- [7]. Khalimsky, E., Kopperman, R., & Meyer, P. R. (1990). Computer graphics and connected topologies on finite ordered sets. *Topology and Its Applications*, 36(1), 1–17.
- [8]. Kong, T. Y., & Rosenfeld, A. (1989). Digital topology: Introduction and survey. *Computer Vision, Graphics, and Image Processing*, 48(3), 357–393.
- [9]. Kovalevsky, V. A. (1989). Finite topology as applied to image analysis. *Computer Vision, Graphics and Image Processing*, 46(2), 141–161.
- [10]. Levine, N. (1963). Semi-open sets and semi-continuity in topological spaces. *The American Mathematical Monthly*, 70(1), 36–41.
- [11]. Levine, N. (1970). Generalized closed sets in topology. *Rendiconti del Circolo Matematico di Palermo*, 19(1), 89–96.
- [12]. Palaniappan, N., & Rao, K. C. (1993). Regular generalized closed sets. *Kyungpook Mathematical Journal*, 33(2), 211–219.
- [13]. Rosenfeld, A. (1979). Digital topology. *The American Mathematical Monthly*, 86(8), 621–630.
- [14]. Serra, J. (1982). *Image analysis and mathematical morphology*. Academic Press.
- [15]. Zomorodian, A., & Carlsson, G. (2005). Computing persistent homology. *Discrete & Computational Geometry*, 33(2), 249–274.