



FORMULAE FOR THREE COMPLEX NUMBERS IN ARITHMETIC PROGRESSION

Mohsin Bashir Parray

Mathematics Educator | Independent Researcher

Srinagar, Jammu & Kashmir, India

Abstract : Let x , y and z be three complex numbers in arithmetic progression with common difference a . This note establishes a general identity relating the powers of these numbers through the relation $x^n + y^n = z^n + K$. Explicit expressions are obtained for the correction term K for positive integers n . The identity follows from the Binomial Theorem and is valid over the field of complex numbers. Applications, special cases and consequences involving power sums, polynomial equations, arithmetic progressions and Diophantine-type problems are presented.

Index Terms — Complex numbers, arithmetic progression, Binomial Theorem, power sums, polynomial equations, correction term.

I. INTRODUCTION

Arithmetic progressions provide a simple structure for studying relationships among numbers separated by a constant difference. Let x , y and z be three complex numbers in arithmetic progression with common difference a . Taking y as the middle term gives $x = y - a$ and $z = y + a$. This note develops an identity for the powers of these three terms and expresses the correction term K explicitly for positive integer powers.

II. MAIN IDENTITIES

Let x , y and z be complex numbers in arithmetic progression with common difference a , so that $x = y - a$ and $z = y + a$. Define $K = x^n + y^n - z^n$, where n is a positive integer.

$$K = (y - a)^n + y^n - (y + a)^n$$

By the Binomial Theorem,

$$(y - a)^n = \sum_{r=0}^n C(n,r) y^{n-r} (-1)^r a^r$$

$$(y + a)^n = \sum_{r=0}^n C(n,r) y^{n-r} a^r$$

Therefore,

$$K = y^n + \sum_{r=0}^n C(n,r) y^{n-r} a^r [(-1)^r - 1]$$

Since $(-1)^r - 1 = 0$ when r is even and -2 when r is odd, only odd powers of a remain. Hence,

$$K = y^n - 2 \sum (C(n,r) y^{n-r} a^r), \text{ for odd } r = 1, 3, 5, \dots, n$$

If n is odd, the largest odd value of r is n . If n is even, the largest odd value of r is $n - 1$. Thus, the identity can be written as $x^n + y^n = z^n + K$, where K is given by the expression above.

III. SPECIAL CASES

From the general formula, the following expressions are obtained:

n	Expression for K
$n = 1$	$K = y - 2a$
$n = 2$	$K = y^2 - 4ya$
$n = 3$	$K = y^3 - 6y^2a - 2a^3$
$n = 4$	$K = y^4 - 8y^3a - 8ya^3$
$n = 5$	$K = y^5 - 10y^4a - 20y^2a^3 - 2a^5$
$n = 6$	$K = y^6 - 12y^5a - 40y^3a^3 - 12ya^5$
$n = 7$	$K = y^7 - 14y^6a - 70y^4a^3 - 42y^2a^5 - 2a^7$

IV. APPLICATIONS AND EXAMPLES

Example 1

If $5^4 + 8^4 = 11^4 + K$, find K . Here, $a = 8 - 5 = 11 - 8 = 3$, $y = 8$ and $n = 4$. Using the formula for $n = 4$:

$$K = y^4 - 8y^3a - 8ya^3$$

$$K = 8^4 - 8(8^3)(3) - 8(8)(3^3) = -9,920$$

Checking the relation gives $625 + 4,096 = 14,641 - 9,920$, or $4,721 = 4,721$.

Example 2

Find three positive integers in arithmetic progression with common difference 2 such that $K = -97$ and $n = 3$. Let y be the middle term. Using $K = y^3 - 6y^2a - 2a^3$ with $a = 2$:

$$-97 = y^3 - 12y^2 - 16$$

$$y^3 - 12y^2 + 81 = 0$$

The source note identifies $y = 3$ as the positive integer solution, giving the terms 1, 3 and 5.

Example 3

Find three consecutive integers x , y and z such that $x^2 + y^2 = z^2$. For $a = 1$ and $n = 2$, $K = y^2 - 4y$. Setting $K = 0$ gives $y(y - 4) = 0$, so $y = 0$ or $y = 4$. The resulting triples are $(-1, 0, 1)$ and $(3, 4, 5)$.

Example 4

For three consecutive integers satisfying $x^3 + y^3 = z^3$, take $a = 1$ and $n = 3$. Then $K = y^3 - 6y^2 - 2$. Setting $K = 0$ gives $y^3 - 6y^2 - 2 = 0$. The source note applies Cardano's method and concludes that there are no three consecutive integers satisfying the equation.

Alternative proof using parity and cube-difference patterns

The source note also presents a parity-based argument. It separates the cases in which x is odd and x is even, uses the stated pattern $1 + 7 + 19 + 37 + 61 + \dots$, and argues by contradiction that no three consecutive positive integers satisfy $x^3 + y^3 = z^3$.

Example 5

For three positive consecutive integers satisfying $x^4 + y^4 = z^4$, take $a = 1$ and $n = 4$. The correction term is $K = y^4 - 8y^3 - 8y$. Setting $K = 0$ gives $y(y^3 - 8y^2 - 8) = 0$. The source note rejects $y = 0$ for positive integers and discusses the roots of the remaining cubic equation.

V. REMARKS AND OBSERVATIONS

- Only odd powers of a appear in the correction term.
- The identity remains valid over the complex field.
- The identity follows directly from the Binomial Theorem.
- The formula provides a systematic way to construct polynomial equations from arithmetic progressions.
- For $n = 2$, the identity leads naturally to Pythagorean-type equations.
- For $n = 3$, the identity leads to cubic Diophantine-type equations.

VI. CONCLUSION

In this note, a general identity has been established for three complex numbers in arithmetic progression. The identity provides an explicit expression for K in the relation $x^n + y^n = z^n + K$. The result follows from the Binomial Theorem and is stated for positive integers n and complex numbers. Several applications and consequences illustrate the usefulness of the identity in algebraic and number-theoretic investigations. Future work may include extending the identity to longer arithmetic progressions and exploring further applications in polynomial and Diophantine equations.

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