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Study of C-Convex Functional

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Abstract :-

The abstract for a C-Convex functional depends heavily on specific context of its application. Generally a C-Convex functional is type of function that exhibits a "Convexity" property with respect to a given set or class of functions, often denoted by "C". This means that the functional satisfies certain condition related to how it behaves under convex combination of functions within the set "C". In this paper we investigate the properties of C-Convex functional in the context of abstract convexity. We define a new class of function C-Convex functional, where "C" is a set of convex functions. Furthermore, we explore the relationship between convex combination of C-set and C-Convex functional.

Introduction :-

We have defined a new type of set C-set. It is the generalization of convex set in a real or complex space.

Definition :-

If A be a non-empty subset of a linear space L. If for $x, y \in A, \alpha x - \beta y \in A$ where $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$, then we define A to be a C-set. Choosing $\alpha = \beta = \frac{1}{2}, y = x$ then $\frac{1}{2}x - \frac{1}{2}x = 0 \in A$; choosing $\alpha = 0, \beta = 1$, we see that $-y \in A$. Also replacing y by $-y$, $\alpha x - \beta(-y) = \alpha x + \beta y \in A$.

Hence a C-set is also a convex set. But it is clear from the definition that a convex set is not necessarily a C-set. Thus a C-set is more general than a convex set. Also 0 is an element of every C-set, which is not true in the case of a convex set. It is also clear that a non-empty set A is a C-set if and only if for $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1, \alpha x \pm \beta y \in A$.

Let L be a linear space on which a real-valued function p is defined, then p is said to be non-negatively homogeneous if $p(\alpha x) = \alpha p(x)$ where $\alpha \geq 0, x \in L$. Also p is called sub-additive if $p(x + y) \leq p(x) + p(y)$.

If $p(0) = 0$ and

$$p\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i p(x_i)$$

for any finite combination with each $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, then p is called a convex functional. Now we will study some theorems.

Theorem (1) :-

All C-Convex functional form a cone of the real linear space of all real-valued functions defined on a linear space L.

Proof :- Let f, g be C-Convex functionals. Let α, β be scalars such that $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$. Thus $f(0) = g(0) = 0$ and for $x, y \in L$ (where the same sign is taken on both sides of the inequality),

$$f(\alpha x \pm \beta y) \leq \alpha f(x) \pm \beta f(y)$$
$$g(\alpha x \pm \beta y) \leq \alpha g(x) \pm \beta g(y)$$

Therefore, $(f + g)(0) = f(0) + g(0) = 0$ and

$$\begin{aligned}(f + g)(\alpha x \pm \beta y) &= f(\alpha x \pm \beta y) + g(\alpha x \pm \beta y) \\ &\leq \alpha f(x) \pm \beta f(y) + \alpha g(x) \pm \beta g(y) \\ &= \alpha f(x) + \alpha g(x) \pm \beta f(y) \pm \beta g(y) \\ &= \alpha(f + g)(x) \pm \beta(f + g)(y)\end{aligned}$$

Thus $f + g$ is also a C-Convex functional.

Let $\lambda \geq 0$, then $(\lambda f)(0) = \lambda f(0) = \lambda \cdot 0 = 0$ and

$$f(\alpha x \pm \beta y) \leq \alpha f(x) \pm \beta f(y)$$

$$\Rightarrow \lambda f(\alpha x \pm \beta y) \leq \lambda\alpha f(x) \pm \lambda\beta f(y)$$
$$\Rightarrow (\lambda f)(\alpha x \pm \beta y) \leq \alpha(\lambda f)(x) \pm \beta(\lambda f)(y)$$

Hence λf is also a C-Convex functional. Therefore, all C-Convex functional form a cone.

Theorem (2) :-

Let f and g be C-Convex functional such that $f : L \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ where L is a linear space and g is increasing, that is, if $x, y \in \mathbb{R}$ such that $x \leq y$ then $g(x) \leq g(y)$; then $g \circ f$, the composition of g with f defined by

$$(g \circ f)(x) = g(f(x))$$

is a C-Convex functional.

Proof :- Since $f : L \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, it follows that $g \circ f : L \to \mathbb{R}$. Let $x, y \in L$ and let α, β be such that $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$. Since f is a C-Convex functional,

$$f(\alpha x \pm \beta y) \leq \alpha f(x) \pm \beta f(y)$$

Since g is increasing and a C-Convex functional (where the same sign is taken on both sides of the inequality),

$$(g \circ f)(\alpha x \pm \beta y) = g(f(\alpha x \pm \beta y)) \leq g(\alpha f(x) \pm \beta f(y))$$
$$\leq \alpha g(f(x)) \pm \beta g(f(y))$$
$$= \alpha(g \circ f)(x) \pm \beta(g \circ f)(y)$$

Also, $(g \circ f)(0) = g(f(0)) = g(0) = 0$. Hence $g \circ f$ is a C-Convex functional.

Theorem (3) :-

Let L be a linear space. Let f, g be C-Convex functionals defined on L, then the function h defined by $h(x) = max\{f(x), g(x)\}, x \in L$ is also a C-Convex functional.

Proof :- Let $x, y \in L$. Since $h(x) = max\{f(x), g(x)\}$, it follows that $f(x) \leq h(x)$ and $g(x) \leq h(x)$. Also $h(y) = max\{f(y), g(y)\} \Rightarrow f(y) \leq h(y)$ and $g(y) \leq h(y)$. Let α, β be such that $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$. Then $\alpha f(x) \leq \alpha h(x)$ and $\beta f(y) \leq \beta h(y)$. Therefore,

$$\alpha f(x) \pm \beta f(y) \leq \alpha h(x) \pm \beta h(y)$$

where the same sign is taken on both sides of the inequality. Similarly, $\alpha g(x) \pm \beta g(y) \leq \alpha h(x) \pm \beta h(y)$. Since f and g are C-Convex functionals, $f(\alpha x \pm \beta y) \leq \alpha f(x) \pm \beta f(y) \leq \alpha h(x) \pm \beta h(y)$ and $g(\alpha x \pm \beta y) \leq \alpha g(x) \pm \beta g(y) \leq \alpha h(x) \pm \beta h(y)$. Therefore,

$$h(\alpha x \pm \beta y) = max\{f(\alpha x \pm \beta y), g(\alpha x \pm \beta y)\} \leq \alpha h(x) \pm \beta h(y)$$

Also, $h(0) = max\{f(0), g(0)\} = max(0, 0) = 0$. Hence h is a C-Convex functional.

References :-

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