THE UPPER PATH INDUCED GEODETIC NUMBER OF SOME GRAPHS

A. Arul Asha[1], S. Joseph Robin[2]
Department of Mathematics, Scott Christain College, Nagercoil, India.

ABSTRACT

Let G be a connected graph. A path induced geodetic set S in a connected graph G is called a *minimal path inducedset* if no proper subset of S is a path induced geodetic set of G. The *upper path inducednumber*$pign^+(G)$is the maximum cardinality of a minimal path induced geodetic *set* of G.Some properties satisfied by this concept arestudied. It isshownfor any positive integers $5 < a \le b$, there exists a connected graph G such that $pign(G) = a$ and $pign^+(G) = b$.

KEYWORDS: geodesic, geodetic number, connected geodetic number, path induced geodeticnumber, path induced geodesicgraphs.upperpath induced geodeticnumber.
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1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and $q$respectively. For basic graph theoreticterminology, werefer to Harary [1].The distance $d(u,v)$betweentwoverticesu and v in a connected graph G is the length of a shortest$u$–$v$path in G. An u–$v$path of length$d(u, v)$ iscalled an u–$v$geodesic. A vertex $x$issaid to lie on a $u - v$geodesic$P$ if $x$isa vertex of Pincluding the vertices$u$ and v. The *eccentricity*$e(v)$ of a vertex $v$in G is the maximum distance from$v$ and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad* Gor$r(G)$and the maximum eccentricityisits*diameter*, *diam* G of G.Twovertices$u$ and v of G are *antipodal* if $d(u, v) = diam\, G$or$d(G)$. A vertex $v$issaid to be an *extremevertex* if the subgraphinduced by itsneighboursiscomplete. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G iscontained in a geodesicjoiningsome pair of vertices in S. The *geodetic number*$g(G)$ of G is the minimum order of its geodetic sets and geodetic set of order $g(G)$ is called $g - set$ or *geodetic basis*. The geodetic number of a graph wasintroducedand studiedin [1,2, 3, 4].A *connectedgeodetic set* of a graph G is a geodetic set S such that the subgraph$G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic set of G is the *connected geodetic number* of G and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called ag_c*-set* of G or a *connected geodetic basis* of G.The connected geodeticnumber of a graph wasintroduced and studiedin [5, 7, 8].A connected geodetic set$S \subseteq V(G)$ is said to be a *path induced geodetic(pig) set of* Gif $< S >$ contais a path $P$with$V(P) = S$.The minimum cardinality of a path induced geodetic set of$G$ is called a *path induced geodetic number* of$G$ and is denoted by $pign(G)$.The path induced geodetic number of a graph wasintroduced inand studied[6].The concept on path-induced geodetic numbers of graphs can be applied in travel time saving, facility location, goods distribution, and other things in which this concept will be of great help.First, we have to remark that not all connected graphs have path-induced geodetic set.Note thatthe pathinducedgeodetic set does not exists for all connected graphs.For exampleatreewith more thantwovertices do not have a the pathinducedgeodetic set. A connected graph G is said to be a *path induced geodesic graph* if G has a path induced geodetic set.The followingtheoremisused in sequel.

Theorem1.1[6]. Eachextreme vertex of a graph connected$G$ belongs to every path induced geodetic set of G. In particular, each end-vertex of G belongs to every path induced geodetic set of G.

2.THE UPPER PATH INDUCED GEODETIC NUBER OF SOME GRAPHS

Definition 2.1. Let G be a connected graph. A path induced geodetic set S in a connected graph G is called a *minimal path inducedset* if no proper subset of S is a path induced geodetic set of G. The*upper path inducednumber*$pign^+(G)$is the maximum cardinality of a minimal path induced geodetic *set* of G.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_2, v_6, v_7, v_8\}, S_2 = \{v_1, v_3, v_6, v_7, v_8\}, S_3 = \{v_1, v_4, v_6, v_7, v_8\}, S_4 = \{v_1, v_5, v_6, v_7, v_8\}$ and $S_5 = \{v_2, v_3, v_5, v_4, v_6, v_7, v_8\}$are the only five minimalpath induced setsof G. Hence $pign^+(G) = 7$.

Remark 2.3. Every minimum path induced set of G is a minimal path induced set of G. The converse is not true. For the graph G given in Figure 2.1, $S_5 = \{v_2, v_3, v_5, v_4, v_6, v_7, v_8\}$ is a minimal path induced setand it is not a minimum path induced set of G.

Theorem 2.4. For a connected graph $G, 2 \le pign(G) \le pign^+(G) \le p$.

Proof. Any path induced geodetic set needs at least two vertices and so $pign(G) \ge 2$. Since every minimum path induced geodetic set is a minimal connected geodetic set, $pign(G) \le pign^+(G)$. Also, since $V(G)$ induces a path induced geodetic set of G, it is clear that $pign^+(G) \le p$. Thus $2 \le pign(G) \le pign^+(G) \le p$. ■

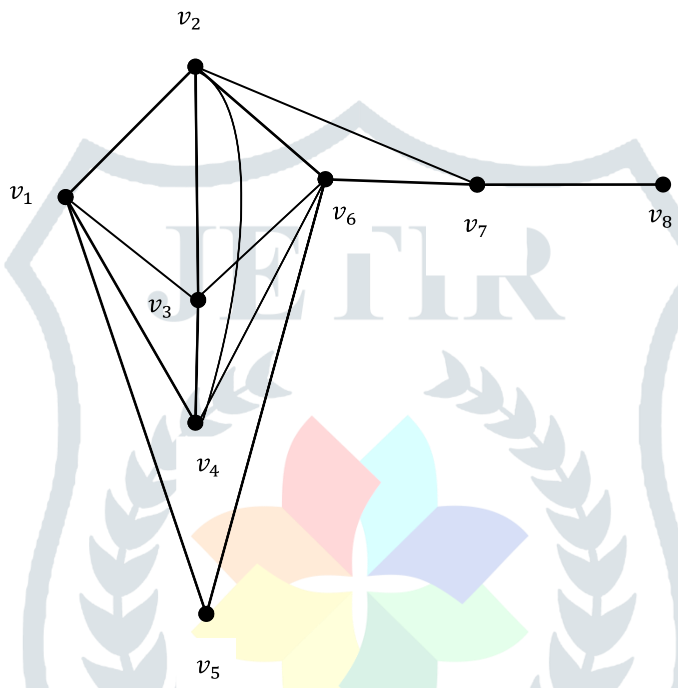
Remark 2.5. For the graph $G = P_2, pign(G) = 2$. For the path $G = P_p, pign^+(G) = p$Also, all the inequalities in Theorem 2.4 are strict. For the graph G given in Figure 2.1, $pign(G)$,$pign^+(G) = 7$, and$p = 8$ so that $2 < pign(G) < pign^+(G) < p$.

Theorem 2.6. Let G be a path induced geodesic graph. Then $pign(G) = p$ if and only if $pign^+(G)$.

Proof. Let $pign^+(G) = p$. Then $S = V(G)$ is the unique minimal path induced geodetic set of G. Since no proper subset of $S$is anpath induced geodetic set, it is clear that S is the unique minimum path induced geodetic set of G andso $pign(G) = p$. The converse follows from Theorem 2.5. ■

Theorem 2.7. Let G be a path induced geodesic graph. Then every extreme vertex of a connected graph G belongs to every minimal path induced geodetic set of G.

Proof. Since every minimal path induced geodetic set is a path induced geodetic set, the result follows from Theorem 1.1. ■



G

Figure 2.1

Theorem 2.8. Let G be a path induced geodesic graph. Thenevery cut-vertex of a connected graph G belongs to every minimal path induced geodetic set of G.

Proof. Let Sbe aminimal path induced geodetic set of $G$and$v$bea cut vertex of G.We prove every component of $G - v$contains an element of S.Suppose thatthereis a component B of $G - v$ such that B contains no vertex of S. Let$u \in V(B)$.Since$S$is a path inducedgeodeticset, thereexists a pair of vertices$x$ and$y$ in Ssuch that u lies on some$x - y$ geodesic $P : x = u_0, u_1, \dots, u, \dots, u_n = y$in G.Since$v$ is a cut-vertex of G, the $x - u$ subpath of P and the $u - y$ subpath of P both contain v, it follows that P is not a path, contrary to assumption. Thereforeevery component of $G - v$contains an element of S. Next we prove that $v \in S$. Let $G_1, G_2, \dots, G_r (r \geq 2)$ be the components of $G - \{v\}$.Since$S$contains at least one elementfromeach$G_i (1 \leq i \leq r)$).Since$< S >$isconnected, itfollowsthat$v \in S$. ■

Corollary 2.9. For a connected graph G with k extreme vertices and l cut-vertices, $pign^+(G) \geq max\{2, k + l\}$.

Proof. This follows from Theorems 2.7 and 2.8. ■

Corollary 2.10. For the complete graph $G = K_p$, $pign^+(G) = p$.

Proof. This follows from Corollary 2.9. ■

Corollary 2.10. For any path $G = P_p$, $pign^+(G) = pign(G) = p$.

Proof. This follows from Corollary 2.9. ■

Theorem 2.11. For the complete bipartite graph $G = K_{m,n} (2 \leq m \leq n)$, $pign^+(G) = 4$.

Proof. Without loss of generality, let $m \leq n$. Let $X = \{x_1, x_2, \dots, x_m\}$,and $Y = \{y_1, y_2, \dots, y_n\}$ be a bipartition of G. Let Sany path induced geodetic set of G. We prove that S contains at least two vertices from$X$ and at least two vertices from$Y$. Suppose that S contains at most one vertex from X and at most one vertex from$Y$. Then S is not a path induced geodetic set of G, which is a contradiction. Therefore S contains at least two vertices from$X$ and at least two vertices from$Y$. We prove that $pign^+(G) = 4$. Suppose $pign^+(G) \geq 4$. Then there exists a minimal path induced geodetic set $S_1$of G with $|S_1| \geq 5$. Since S_1 contains at least two vertices from X and at least two vertices from Y, without loss of generality, let $x_1, x_2, x_3, y_1, y_2 \in S_1$. Then $S_2 = S_1 - \{x_3\}$ is a path induced geodetic set of G with $S_2 \subset S_1$, which is a contradiction to S_1 a minimal path induced geodetic set of$G$.Thus $pign^+(G) = 4$. ■

Theorem 2.12. For the cycle $= C_p$,$pign^+(G) = \begin{cases} \frac{p}{2}+1 & \text{if } p \text{ is even} \\ \left\lfloor \frac{p}{2} \right\rfloor + 2 & \text{if } p \text{ is odd.} \end{cases}$

Proof.

Case 1. Suppose that p is even. Let $p = 2n$. Let $C_{2n}: v_1, v_2, v_3, \ldots, v_{2n}, v_1$be the cycle of order $2n$. Let $S = \{v_1, v_2, v_3, \ldots, v_{n+1}\}$.Then$S$ is a connected geodetic set of G and $P: v_1, v_2, v_3, \ldots, v_{n+1}$ is a path in $< S >$with $V(P) = S$. Therefore S is a path induced geodetic set of$G$.We prove that S is a minimal path induced geodetic set of G.Since v_{n+1} is the antipodal vertex of v_1, $\{v_1, v_{n+1}\}$ is a geodetic set of G. Since no other elements of S are antipodal, there is no subset of S is a path induced geodetic set o G.Hence it follows that S is a minimal path induced geodetic set of $G$and so $pign^+(G) \geq |S| = n + 1 = \frac{p}{2} + 1$.

Now, we showthat $pign^+(G) = n + 1$. Otherwise, there is a minimal path induced geodetic set et W such that$|W| = m > n + 1$. Since$W$ is a path induced geodetic set of G, the subgraph induced by$W$is a path say $v_{i+1}, v_{i+2}, \ldots, v_{i+m}$. It is clear that $T = \{v_{i+1}, v_{i+2}, \ldots, v_{i+m+1}\}$ is a path induced geodetic set of G and so W is not a minimal connected geodetic set of G, which is a contradiction. Thus $pign^+(G) = n + 1$

Case 2. Suppose that p is odd and let $p = 2n + 1$.Let$C_{2n+1}: v_1, v_2, v_3, \ldots, v_{2n+1}, v_1$be the cycle of order $2n + 1$. Let $S = \{v_1, v_2, v_3, \ldots, v_{n+2}\}$. Then, as in Case 1 it is seen that S is a minimal path induced geodetic set of $G$and $pign^+(G) = n + 2 = \left\lfloor \frac{p}{2} \right\rfloor + 2$. ■

Theorem 2.13. For the wheel$G = K_1 + C_{p-1}, (p \geq 4), pign^+(G) = p - 2$.

Proof : Let C_{p-1} be a $v_1, v_2, \ldots, v_{p-1}$ and x be the vertex of K_1. Then x is a vertex of degree $p - 1$. The pig-set of $G$is$S_i = V(G) - \{x, v_i\} (1 \leq i \leq p - 1)$ so that $pign(G) = p - 2$. We prove that $pign^+(G) \geq p - 2$. By Theorem 2.6, $pign^+(G) = p - 1$. Let Sbe a minimal path induced geodetic setof $G$with $|p - 1|$. If $x \in S$, $S$is not a pathinducedgeodetic set of G.If $x \in S$, then $S_1 = S - \{y\}$,where $y \neq x$is a path induced geodetic setof G with $S_1 \subset S$.Whichis a contradiction to Sa minimal path induced geodetic setof G.Therefore$pign^+(G) = (G) = p - 2$.■

Theorem 2.14. For any positive integers $5 < a \leq b$, there exists a connected graph G such that $pign(G) = a$ and $pign^+(G) = b$.

Proof. If $a = b$, let $G = P_p$. Then by Corollary 2.10, $pign(G) = a = pign^+(G)$.

Let $5 < a < b$ .Let $P_a: u_1, u_2, \ldots, u_a$ be a path of length $a - 1$. Let $H = K_{b-a+1}$ and $V(H)$ be $v_1, v_2, \ldots, v_{b-a+1}$. Let G be the graph obtained from P_a and $V(H)$ by joining u_1 and u_3 with each $v_i (1 \leq i \leq b - a + 1)$, there by producing the graph G of Figure 2.2.First we show that $pign(G) = a$. Let $S = \{u_3, u_4, \ldots, u_a\}$ be the set of all cut vertices and extreme vertices of G. By Theorems 2.6 and2.8, every path induced geodetic set of G contains S. It is clear that S is not a path induced geodetic set of G.It is easily verified that $S \cup \{x\}$, where $x \notin S$ is not a path induced geodetic set of $G$and so $pign(G) \geq a$. Let $S_1 = S \cup \{u_1, u_2\}$.Then $S_1$is a path induced geodetic set of G so that $pign(G) = a$.Next we show that $pign^+(G) = b$.

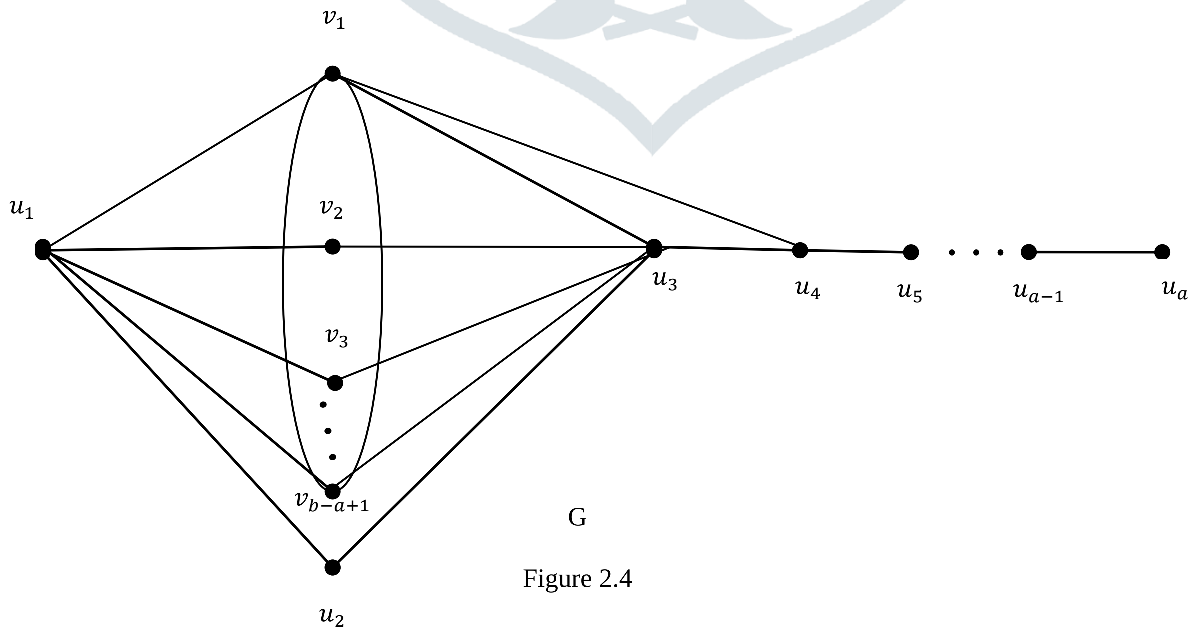


Figure 2.4

The pig-sets of G are $S_i = S \cup \{u_1\} \cup \{v_i\} (1 \leq i \leq b - a + 1)$ and $S_1 = S \cup \{u_1, u_2\}$Let $W = S \cup \{v_1, v_2, \ldots, v_{b-a+1}, u_2\}$. It is clear that $W$is a path induced geodetic set of G. Now, we show that W is a minimal is a path induced geodetic set of G. Assume, to the contrary, that W is not a minimal is a path induced geodetic set of G. Then there is a proper subset T of W such that T is a path induced geodetic set of G. Let $v \in W$ and $v \notin T$. By Theorems 2.6 and 2.8, it is clear$v = u_2$ or$v = v_i$, for some $i = 1, 2, \ldots,$ $b - a + 1$.Clearly, this $v$does not lie on a geodesic joining any pair of vertices of T and so T is not a path induced geodetic set of G, which is a contradiction. Thus S_2 is a is a path induced geodetic set of $G$and so $pign^+(G) \geq b$ Since the order of the graph is b, it follows that $pign^+(G) = b$.

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