CLASSIFICATION OF TOTALLY UMBILICAL SLANT SUBMANIFOLDS OF A (k,μ) - CONTACT MANIFOLD

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Abstract: The objective of this paper is to classify totally umbilical slant submanifolds of a (k, μ)-contact manifold. We prove that a totally umbilical slant sub-manifold M of a (k, μ)-contact manifold $\overline{M}$ is either invariant or anti-invariant or dim M=1 or the mean curvature vector H of M lies in the invariant normal sub bundle.

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1. INTRODUCTION

Slant Submanifolds has prominence in the field of submanifolds because of invariant and anti-invariant conditions. Chen [3] introduced the slant submanifold for an almost Hermitian manifold, which includes generalisation of both holomorphic and totally real submanifolds. He studied regarding the classification of totally umbilical submanifolds in symmetric spaces [4]. Chen [2] discussed Slant submanifolds. Lotta [6,7] introduced the notion of slant immersions where he has obtained the results of fundamental importance. The geometry of slant submanifold is studied by Cabrerizo et al. [1, 11] in more specialized settings of K-contact and sasakian manifolds, while slant submanifolds of a Kaehler manifold were given by Maeda et al. [8]. Later Gupta [5] et al. defined and studied about Slant submanifold of a Kenmotsu manifold. Siddesha and Bagewadi [9] studied about (k,μ)-contact manifolds. In 1954, Schouten studied the totally umbilical submanifolds and proved that every totally umbilical submanifold of dim≥ 4 in a conformally flat space is conformally flat [10].

In this paper, we consider M, a totally umbilical slant submanifold tangent to the structurevector field of a (k, μ)-contact manifold $\overline{M}$ and obtain a classification result that either(i) M is anti-invariant or (ii) dim M=1 or (iii) H $\epsilon\Gamma(\mu)$, where μ is the invariant normal subbundle under φ. We also prove that every totally umbilical proper slant submanifold is totally geodesic

2. PRELIMINARIES

Let ($\overline{M}$,g)be an almost contact metric manifold of dimension (2n + 1) equipped with structure (φ,ξ,η,g) consisting of a (1,1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying,

$$\varphi^2X = -X + \eta(X)\xi,\eta(\xi) = 1,\varphi\xi = 0,\eta(\varphi X) = 0, \quad (2.1)$$

$$g(\varphi X,\varphi Y) = g(X,Y)-\eta(X)\eta(Y),\eta(X) = g(X,\xi), g(X,\varphi Y) = -g(\varphi X,Y). \quad (2.2)$$

for any X,Y tangent to $\overline{M}$. An almost contact metric manifold is called (k, μ) contact manifold if

$$(\overline{\nabla}_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX), \quad (2.3)$$

$$(\overline{\nabla}_X\xi) = -\varphi X - \varphi hX \quad (2.4)$$

where $\overline{\nabla}$ denotes the Levi-civita connection on $\overline{M}$.

Let ∇ be the Riemannian metric induced on M which is denoted by the same symbol g, then the Gauss and Weingarten formulae are given by,

$$\overline{\nabla}_XY = \nabla_XY + h(X,Y), \quad (2.5)$$

$$\overline{\nabla}_XN = \nabla^{\perp}_XN - A_NX, \quad (2.6)$$

for any X,Y ∈ TM and N ∈ $T^{\perp}M$ of $\overline{M}$.

$\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}M$ of M, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by,

$$g(h(X,Y),N) = g(A_NX,Y). \quad (2.7)$$

Let M be a n-dimensional Riemannian manifold with induced metric g isometrically immersed in $\bar{M}$. We denote by TM, the Lie algebra of vector fields on M and by$T^{\perp}M$,the set of all vector fields normal to M.

For X $\in\Gamma$ (TM) and N $\in\Gamma$ ($T^{\perp}M$), we write

$\varphi X = TX + NX,$ (2.8)

$\varphi V = tV + nV,$ (2.9)

where TX and NX denotes the tangential and normal component of φX. Similarly tV and nV denotes the tangential and normal component of φV .

If T is the endomorphism defined by (2.5) then,

$g(TX,Y) = -g(X,TY).$ (2.10)

The covariant derivatives of the endomorphisms φ, T and F are defined respectively as

$(\bar{\nabla}_X\varphi)Y = \bar{\nabla}_X\varphi Y - \varphi\bar{\nabla}_X Y, \forall X,Y \in \Gamma(T\bar{M})$ (2.11)

$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \forall X,Y \in \Gamma(TM)$ (2.12)

$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \forall X,Y \in \Gamma(TM)$ (2.13)

Throughout, the structure vector field ξ is assumed to be tangential to M. Otherwise M is simply anti-invariant.
For any X ∈ Γ(TM), on using equations (2.4) and (2.5), we get,

(a) $(\nabla X\xi) = -\varphi X - \varphi hX,$ (b) $h(X,\xi) = 0.$ (2.14)

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be totally umbilical if

$h(X,Y) = g(X,Y)H,$ (2.15)

where H is the mean curvature vector of M.

Further if h(X,Y) = 0, ∀X,Y ∈ Γ(TM), then M is said to be totally geodesic and if
H = 0, then M is minimal in $\bar{M}$. For totally umbilical submanifold M tangent to the structure vector field ξ of a (k,μ) contact manifold $\bar{M}$, we have,

$g(X,\xi)H = 0, \forall X \in \Gamma(TM)$ (2.16)

3. SLANT SUBMANIFOLD OF (k,μ)-CONTACT MANIFOLD

A submanifold M tangent to the structure vector field ξ of an almost contact metric manifold $\bar{M}$ is said to be slant submanifold if for any x ∈ M and X ∈ TxM −<ξ>, the angle between φX and TxM is constant. The constant angle θ ∈ [0,π/2] is then called slant angle of M in $\bar{M}$. Thus, for a slant submanifold M, the tangent bundle TM is decomposed as

$TM = D\oplus<\xi>,$ (3.1)

where the orthogonal complementary distribution D of <ξ> is known as slant distribution on M. The normal bundle $T^{\perp}M$ of M is discomposed as

$T^{\perp}M = F(TM)\oplus\mu,$ (3.2)

where μ is the invariant normal subbundle with respect to φ orthogonal to F(TM). For a proper slant submanifold M of an almost contact metric manifold $\bar{M}$ with the slant angle θ, Lotta[8] proved that

$T^2X = -\cos^2\theta[X - \eta(X)\xi],$ (3.3)

for any X ∈ Γ(TM).

From [7] we know that If M is a submanifold of an almost contact metric manifold $\bar{M}$ such that ξ ∈ TM. Then, M is slant if and only if there exists a constant λ ∈ [0,1] suchthat

$T^2(X) = -\lambda(X - \eta(X)\xi).$ (3.4)

Further more, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.
Also for any X,Y ∈ TM, we have,

$g(TX,TY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)),$ (3.5)

$g(FX,FY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)).$ (3.6)

for any X,Y ∈ Γ(TM).

In the following theorem we consider M as a totally umbilical slant submanifold of a (k,μ) contact manifold $\bar{M}$.

Theorem 3.1:

If M is a totally umbilical slant submanifold of (k,μ)-contact manifold $\bar{M}$ such that

$(\varphi h)X = -h(\varphi X),$ (3.7)

Then

$Th = -hT$ and $Fh = -hF$ (3.8)

Proof: As M is a n-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$, from (2.8) we have,

$\varphi X = TX + FX,$

we have,

$(\varphi h)X = ThX + FhX,$ (3.9)

and

$h(\varphi X) = h(TX) + h(FX).$ (3.10)

If $(\varphi h)X = h(\varphi X)$, then we have,

$ThX + FhX = -[h(TX) + h(FX)].$ (3.11)

Equating the tangential and normal components, we get,

$ThX = -h(TX)$ and $FhX = -h(FX).$ (3.12)

Implies,

$Th = -hT$ and $Fh = -hF.$

Hence the proof.

Theorem 3.2.:

Let M be a totally umbilical slant submanifold of a (k,μ)-contact manifold $\overline{M}$ .Then atleast one of the following statements is true.

(i) M is invariant.

(ii) M is anti-invariant.

(iii) M is totally geodesic.

(iv) dimM = 1.

(v) If M is proper slant, then $H \in \Gamma(\mu)$.

where H is the mean curvature vector of M.

Proof: As M is totally umbilical slant submanifold, we have,

$$h(TX,TX) = g(TX,TX)H = Cos^2\theta[\|X\|^2 - \eta^2(X)]H. \quad (3.13)$$

Using equation (2.5), we obtain,

$$Cos^2\theta[\|X\|^2 - \eta^2(X)]H = \overline{\nabla}_{TX}TX - \nabla_{TX}TX. \quad (3.14)$$

If ξ is orthonormal to vector X then g(X,ξ) = 0 and also from (2.8), we get,

$$Cos^2\theta[\|X\|^2]H = \overline{\nabla}_{TX}\varphi X - \overline{\nabla}_{TX}FX - \nabla_{TX}TX. \quad (3.15)$$

Now by (2.6) and (2.11) we derive,

$$Cos^2\theta[\|X\|^2]H = (\overline{\nabla}_{TX}\varphi)X + \varphi(\overline{\nabla}_{TX}X) + A_{FX}TX - \nabla^{\perp}_{TX}FX - \nabla_{TX}TX, \quad (3.16)$$

using (2.3) and (2.5), we obtain

$$Cos^2\theta[\|X\|^2]H = g(TX,X)\xi + g(hTX,X)\xi + \varphi[\nabla_{TX}X + h(TX,X)] + A_{FX}TX - \nabla^{\perp}_{TX}FX - \nabla_{TX}TX. \quad (3.17)$$

From (2.8),(2.10) and (2.15) also the fact that X and TX are orthogonal vector fields on M, we arrive at

$$Cos^2\theta[\|X\|^2]H = g(hTX,X)\xi + T\nabla_{TX}X + F\nabla_{TX}X + A_{FX}TX - \nabla^{\perp}_{TX}FX - \nabla_{TX}TX. \quad (3.18)$$

Using (3.4) and (3.6) we get,

$$Cos^2\theta[\|X\|^2]H = -Cos^2\theta[g(hX,X)]\xi + T\nabla_{TX}X + F\nabla_{TX}X + A_{FX}TX - \nabla^{\perp}_{TX}FX - \nabla_{TX}TX. \quad (3.19)$$

Taking inner product with TX in (3.19) for any $X \in \Gamma(TM)$, We obtain,

$g(T\nabla_{TX}X,TX) + g(A_{FX}TX,TX) - g(\nabla_{TX}TX,TX) = 0.$ (3.20)

Now, we compute the first and last term of (3.20) as follows

$g(T\nabla_{TX}X,TX) = Cos^2\theta[g(\nabla_{TX}X,X) - \eta(X)g(\nabla_{TX}X,\xi)].$

If ξ is orthogonal to vector X, then g(X,ξ) = 0, therefore

$g(T\nabla_{TX}X,TX) = Cos^2\theta[g(\nabla_{TX}X,X)].$ (3.21)

Also, we have,

$g(\nabla_{TX}TX,TX) = g(\nabla_{TX}TX,TX).$

Using the property of Riemannian connection the above equation will be,

$g(\nabla_{TX}TX,TX) = 1/2[TXg(TX,TX)]$,
$= 1/2[TX(Cos^2\theta g(X,X)-\eta(X)^2)]$.
Again by the property of Riemannian connection, we derive

$g(\nabla_{TX}TX,TX) = Cos^2\theta[g(\overline{\nabla}_{TX}X,X)-\eta(X)g(\overline{\nabla}_{TX}X,\xi)]-Cos^2\theta\eta(X)g(\overline{\nabla}_{TX}\xi,X)$. (3.22)

If ξ is orthogonal to vector X, then $g(X,\xi) = 0$ and the fact that X and TX are orthogonal vector fields on M, the last term of (3.22) is identically zero, then by (2.5) we obtain,

$$g(\nabla_{TX}TX,TX) = Cos^2\theta[g(\nabla_{TX}X,X)]. \quad (3.23)$$

Thus, from (3.21) and (3.23), we get,

$$g(T\nabla_{TX}X,TX) = g(\nabla_{TX}TX,TX). \quad (3.24)$$

Using this fact (3.24) in (3.20) we obtain,

$$g(A_{FX}X,TX) = g(h(TX,TX),FX) = 0 \quad (3.25)$$

As M is totally umbilical slant, from (2.15) and (3.5), we get,

$$0 = Cos^2\theta[\|X\|^2]g(H,FX) \quad (3.26)$$

Thus, from(3.26) we conclude that either $\theta = \pi/2$, that is M is anti-invariant which part (ii) or the vector field X is parallel to the structure vector field ξ, i.e., M is 1-dimensional submanifold which is fourth part of the theorem or $H \perp FX, \forall X \in \Gamma(TM)$, i.e., $H \in \Gamma(\mu)$ which is the last part of the theorem or $H = 0$ i.e., M is totally geodesic which is (iii) or $FX = 0$, for all $X \in \Gamma(TM)$, i.e., M is invariant which is part (i).

This proves the theorem completely.

Now, if we consider M, a proper slant submanifold of a (k,μ) contact manifold $\overline{M}$, then neither M is invariant nor anti-invariant (by definition of proper slant) and also neither dimM = 1. Hence, by the above result, only possibility is that $H \in \Gamma(\mu)$ for a totally umbilical proper slant submanifold. Thus, we prove the following main result.

Theorem 3.3.
Every totally umbilical proper slant submanifold of a (k,μ) contact manifold is totally geodesic.

Proof: Let M be a totally umbilical proper slant submanifold of a (k,μ) contact manifold $\overline{M}$, then for any $X,Y \in (\Gamma(TM))$, we have from (2.3)

$$\overline{\nabla}_X\varphi Y - \varphi\overline{\nabla}_X Y = g(X + hX,Y)\xi - \eta(Y)(X + hX),$$

from (2.5) and (2.8) we obtain,

$$\overline{\nabla}_X TY + \overline{\nabla}_X FY - \varphi[\nabla_X Y + h(X,Y)] = g(X,Y)\xi + g(hX,Y)\xi - \eta(Y)X - \eta(Y)hX.$$

Again using (2.5), (2.6) and (2.8) we get,

$$\nabla_X TY + h(X,TY) + \nabla^{\perp}_X FY - A_{FY}X - T[\nabla_X Y + h(X,Y)] - F[\nabla_X Y + h(X,Y)] = g(X,Y)\xi + g(hX,Y)\xi - \eta(Y)X - \eta(Y)hX.$$

As M is totally umbilical, then

$$\nabla_X TY + g(X,TY)H + \nabla^{\perp}_X FY - A_{FY}X - T\nabla_X Y - Th(X,Y) - F\nabla_X Y - Fh(X,Y) = g(X,Y)\xi + g(hX,Y)\xi - \eta(Y)X - \eta(Y)hX.$$

If ξ is orthogonal to vector Y, then $g(Y,\xi) = 0$, therefore

$$\nabla_X TY + g(X,TY)H + \nabla^{\perp}_X FY - A_{FY}X - T\nabla_X Y - F\nabla_X Y - g(X,Y)\varphi H = g(X,Y)\xi + g(hX,Y)\xi. \quad (3.27)$$

Taking the inner product with φH in (3.27) and using the fact that $H \in \Gamma(\mu)$, we obtain

$$g(\nabla^{\perp}_X FY,\varphi H) = g(X,Y)\|H\|^2 \quad (3.28)$$

Using (2.6) and the property of Riemannian connection, the above equation takes the form

$$g(\nabla^{\perp}_X\varphi H,FY) = -g(X,Y)\|H\|^2 \quad (3.29)$$

Now, for any $X \in \Gamma(TM)$ we have,

$$\overline{\nabla}_X\varphi H = (\overline{\nabla}_X\varphi)H + \varphi\overline{\nabla}_X H \quad (3.30)$$

Using (2.3), (2.6) and (2.8) and the fact that $H \in \Gamma(\mu)$ we obtain,

$$-A_{\varphi H}X + \nabla^{\perp}_X\varphi H = -TA_H X - FA_H X + \varphi\nabla^{\perp}_X H \quad (3.31)$$

Also for any $X \in \Gamma(TM)$, we have,

$g(\nabla^{\perp}_X H,FX) = g(\overline{\nabla}_X H,FX)$,
$= -g(H,\overline{\nabla}_X FX)$

Using (2.8), we get,

$$g(\nabla^{\perp}_X H,FX) = -g(H,\overline{\nabla}_X\varphi X) + g(H,\overline{\nabla}_X PX). \quad (3.32)$$

Then from (2.5) and (2.11), we derive,

$g(\nabla^{\perp}_X H, FX) = -g(H, (\overline{\nabla}_X \varphi)X) - g(H, \varphi\overline{\nabla}_X X) + g(H, h(X, PX)).$ (3.33)

Using (2.3) and (2.15), the first and last term of right hand side of the above equation are identically zero and hence by (2.2), the second term gives

$g(\nabla^{\perp}_X H, FX) = g(\varphi H, \overline{\nabla}_X X).$

Again using (2.5) and (2.15), finally we obtain,

$g(\nabla^{\perp}_X H, FX) = g(\varphi H, H)\|H\|^2 = 0.$

Implies

$\nabla^{\perp}_X H \in \Gamma(\mu).$ (3.34)

Now, taking the inner product in (3.31) with FY , for any $Y \in \Gamma(TM)$, we get,

$g(\nabla^{\perp}_X \varphi H, FY) = -g(FA_H X, FY) + g(\varphi\nabla^{\perp}_X H, FY).$ (3.35)

Using (3.34), the last term of the right hand side of the above equation will be zero and then from (3.6) and (3.29) we obtain,

$g(X,Y)\|H\|^2 = \sin^2\theta[g(X,Y)\|H\|^2].$ (3.36)

Therefore, the above equation can be written as

$\cos^2\theta g(X,Y)\|H\|^2 = 0.$ (3.37)

Since M is proper slant, thus from (3.37), we conclude that H = 0 i.e., M is totally geodesic in $\overline{M}$.

This completes the proof the theorem.

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