A New Type of Quasi- open and Quasi - closed functions in Topological Spaces

[1] Reena C, [2] Vijayalakshmi P, [3] S. Pious Missier
[1] Assistant Professor, PG and Research Department of Mathematics, St.Mary's College,Tuticorin.
Affliated to Manonmaniam Sundaranar University, Tirunelveli, India
[2] M.Phil Scholar, PG and Research Department of Mathematics, St.Mary's College,Tuticorin.
Affliated to Manonmaniam Sundaranar University, Tirunelveli, India
[3] Associate Professor (Retd), PG and Research Department of Mathematics, V.O.C College,Tuticorin.
Affliated to Manonmaniam Sundaranar University, Tirunelveli, India

***Abstract:** The purpose of this paper is to introduce the concepts of Quasi semi*δ-open and Quasi semi*δ-closed functions using semi*δ-open sets and investigate their basic properties. We also discuss their relationships with already existing concepts.*

***AMS Subject Classification:** 54C10, 54C08, 54C05.*

***Keywords:** semi*δ -open set, semi*δ -closed set, quasi semi*δ -open function and quasi semi*δ-closed function.*

I INTRODUCTION

Levine [2] offered a new and useful notion in General Topology that is the notion of a generalized closed set. This notion has been studied extensively in recent years by many topologists. The investigation of generalized closed sets has led to several new and interesting concepts. After the introduction of generalized closed sets there are many research papers which deal with different types of generalized closed sets. Dunham[1] additionally defined a new closure operator cl* by using generalized closed sets. Quite recently S.Pious Missier and C.Reena [4] introduced a new notion of generalized closed sets called semi*δ-closed sets.

In this paper, we will continue the study of related functions by involving semi*δ-open set. We introduce and characterize the concept of quasi semi*δ-open and quasi semi*δ- closed functions.

II PRELIMINARIES

Throughout this paper (X, τ), (Y, σ) and (Z, η) will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X, τ), cl (A) and int (A) denote the closure and the interior of A respectively. We recall some known definitions needed in this paper.

Definition 2.1: A subset A of a topological space (X, τ) is called a **semi*δ-open set** [4] if there exists a δ-open set U in X such that U⊆A⊆Cl*(U).

Definition 2.2: The **semi*δ-interior** [4] of A is defined as the union of all semi*δ-open sets of X contained in A. It is denoted by **s*δInt(A).**

Definition 2.3: A is **semi*δ-closed** [5] if Int*(δCl(A))⊆A.

Definition 2.4: The **semi*δ-closure** [5] of A is defined as the intersection of all semi*δ-closed sets in X containing A. It is denoted by **s*δCl(A).**

Definition 2.5: A function f: X→ Y is said to be **semi*δ-continuous** [6] if $f^{-1}(V)$ is semi*δ-open in (X, τ) for every open set V in (Y, σ).

Definition 2.6: A function f: X⟶Y is said to be **semi*δ-irresolute** [6] if $f^{-1}(V)$ is semi*δ-open in X for every semi*δ-open set V in Y.

Definition 2.7: A function f: X⟶Y is said to be **semi*δ-open** [7] if f (U) is semi*δ-open in Y for every open set U in X.

Definition 2.8: A function f: X⟶Y is said to be **pre-semi*δ-open** [7] if f(U) is semi*δ-open in Y for every semi*δ-open set U in X.

Definition 2.9: A function f: X⟶Y is said to be **semi*δ-closed** [7] if f(F) is semi*δ-closed in Y for every closed set F in X.

Definition 2.10: A function f: X⟶Y is said to be **pre-semi*δ-closed** [7] if f(F) is semi*δ-closed in Y for every semi*δ-closed set F in X.

III QUASI SEMI* δ - OPEN FUNCTIONS

Definition 3.1: A function $f:(X,\tau)\to(Y,\sigma)$ is said to be **quasi semi*δ- open** if the image of every semi* δ-open set in X is open in Y.

Example3.2: Let X =Y= {a, b, c}, $\tau=\{\emptyset,\{a\},\{b\},\{a,b\},X\}$ and $\sigma=\{\emptyset,\{a\},\{c\},\{a,c\},X\}$. In this space $S^*\delta O(X,\tau)=\{X,\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{c,a\}\}$ and $S^*\delta O(Y,\sigma)=\{X,\emptyset,\{a\},\{c\},\{a,b\},\{b,c\},\{c,a\}\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = f(c) = c. Clearly, f is quasi S*δ open

Theorem 3.3: A function $f:(X,\tau)\to(Y,\sigma)$ is quasi semi*δ-open if and only if for every subset U of X, $f(s^*\delta\text{-int}(U))\subset \text{int}(f(U))$.

Proof: Let f be quasi semi*δ-open function. Now we have int (U) ⊂ U and S*δ-int(U) is a semi*δ – open set. Hence we obtain that, s*δ-int(U)) ⊂f (U). As f(s*δ-int(U)) is open ,then f(s*δ-int (U)) ⊂ int (f(U)).

Conversely, Assume that U is a semi*δ-open set in X. Then f(U) =f(s*δ- int(U)) ⊂ int (f(U)) but int (f(U)) ⊂ f(U). Consequently, f(U) = int(f(U)) and f is quasi semi*δ –open.

Lemma 3.4: If a function $f:(X,\tau)\to(Y,\sigma)$ is quasi semi*δ-open, then $s^*\delta\text{-int}(f^{-1}(V))\subset f^{-1}(\text{int}(V))$ for every subset V of Y.

Proof: Let V be any arbitrary subset of Y then s*δ int $(f^{-1}(V))$ is a semi*δ- open set in X and f is quasi semi*δ-open.

Then $f(s^*\delta\text{-int}(f^{-1}(V)) \subset \text{int}(f(f^{-1}(V))$

$\subset \text{int}(V)$

$s^*\delta\text{-int}(f^{-1}(V)) \subset f^{-1}(\text{int}(V))$.

Definition 3.5: A subset S is called a semi*δ-neighbourhood of a point x of X if there exists a semi*δ- open set U such that x ∈ U ⊂ S.

Theorem 3.6: Let function $f:(X,\tau)\to(Y,\sigma)$ be a function, then the following are equivalent.

(i) f is quasi semi*δ -open function

(ii) for each subset U of X , $f(s^*\delta-\text{Int}(U))\subset \text{int}(f(U))$

(iii) for each x ∈ X and each semi*δ-neighbourhood U of x in X, there exists a neighbourhood V of f(x) in Y such that V ⊂ f(U).

Proof: **(i)** ⟹ **(ii)** It follows from the theorem 3.3.

(ii) ⟹ **(iii)**: Let $x\in X$ and U be an arbitrary semi*δ-neighbourhood of x in X. then there exists a semi*δ- open set V in X such that $x\in V\subset U$. Then by (ii) we have $f(V)=f(s^*\delta\, Int\,(V))\subset Int\,(f(V))$ and hence $f(V)=Int\,f(V)$. Therefore, it follows that f(V) is open set in Y such that $f(x)\in f(V)\subset f(U)$.

(iii) ⟹ **(i)**: Let U be an arbitrary semi*δ- open set in X. Then for each $y\in f(U)$, by (iii) there exists a neighbour hood V_y of y in Y such that $V_y\subset f(U)$. As V_y is a neighbourhood of y, there exists an open set W_y in Y such that $y\in W_y\subset V_y$. Thus, $f(U)=\cup\{W_y : y\in f(U)\}$ which is a open set in Y which implies f is a quasi semi*δ – open function.

Theorem 3.7: A function $f:(X,\tau)\to(Y,\sigma)$ is quasi semi*δ- open function iff for any subset B of Y and for any semi*δ- closed set F of X containing $f^{-1}(B)$, there exists a closed set G of Y containing B such that $f^{-1}(G)\subset F$

Proof: Suppose f is a quasi semi*δ-open function. Let $B\subset Y$ and F be a semi*δ-closed set containing $f^{-1}(B)$. Put $G=Y-f(X-F)$. It is clear that $f^{-1}(B)\subset F$ implies $B\subset G$. Since f is quasi semi*δ-open. We obtain, G as a closed set of Y. Moreover, we have $f^{-1}(G)\subset F$.

Conversely, let U be an semi*δ-open set of X and put $B=Y-f(U)$.Then $X-U$ is a semi*δ-closed set in X containing $f^{-1}(B)$.By hypothesis, there exists a closed set F of Y such that $B\subset F$ and $f^{-1}(F)\subset(X-U)$. Hence, we obtain $f(U)\subset(Y-F)$. On the other hand, it follows that $B\subset F$, $Y-F\subset Y-B=f(U)$. Thus, we obtain $f(U)=Y-F$ which is open and hence, f is a quasi semi*δ- open function.

Theorem 3.8: A function $f:(X,\tau)\to(Y,\sigma)$ is a quasi semi*δ-open function iff $f^{-1}\left(cl(B)\right)\subset s^*\delta\, cl\,(f^{-1}(B))$ for every subset B of Y.

Proof: Suppose that f is quasi semi*δ –open. Then for any subset B of Y, $f^{-1}(B) \subset s^{*}\delta\, cl(f^{-1}(B))$. Therefore by theorem 3.7, there exists a closed set F in Y such that $B \subset F$ and $f^{-1}(F) \subset s^{*}\delta\, cl\left(f^{-1}(B)\right)$. Hence we obtain $f^{-1}\left(cl(B)\right) \subset f^{-1}(F) \subset s^{*}\delta cl\, (f^{-1}(B))$.
Conversely, let $B \subset Y$ and F be semi*δ- closed set of X containing $f^{-1}(B)$. Put W = $cl_Y\,(B)$, then we have $B \subset W$ and W is a closed and $f^{-1}(W) \subset s^{*}\delta\, cl\left(f^{-1}(B)\right) \subset F$. Then by theorem 3.7, f is quasi semi*δ –open.

Theorem 3.9: Let $f{:}\,X \to Y$ and $g{:}\,Y \to Z$ be function and $g \circ f{:}\,X \to Z$ is a quasi semi*δ- open. If g is continuous injective, then f is quasi semi*δ-open.

Proof: Let U be a semi*δ-open set in X, Since $(g \circ f)$ is quasi semi*δ- open, $g \circ f\ (U)$ is open in Z. Again g is injective continuous function, f(U) = $g^{-1}\left(g \circ f(U)\right)$ is open in Y. This shows that f is quasi semi*δ- open.

Theorem 3.10: Let f: X → Y and g: Y → Z be any two functions. Then the following properties hold:

(i) If f is semi*δ-open and g is quasi semi*δ-open function, then g ∘ f is open function.

(ii) If f is quasi semi*δ-open and g is semi*δ-open, then g ∘ f is pre semi*δ-open function.

(iii) If f is pre-semi*δ- open function and g is quasi semi*δ-open, then g ∘ f is quasi semi*δ- open.

Proof:

(i) Let V be any open set in X. Since,f is semi*δ- open function, f(V) is a semi*δ-open in Y. Since g is a quasi semi*δ-open, g(f(V)) is open set in Z. That is , g ∘ f(V) = g(f(V)) is open in Z and hence g ∘ f is open function

(ii) Let V be any semi*δ-open set in X. Since, f is quasi semi*δ-open, f(V) is open in Y. Since, g is semi*δ- open function, g(f(V)) is semi*δ- open in Z. That is g ∘ f(V) = g(f(V)) is semi*δ-open in Z and hence g ∘ f is pre-semi*δ-open.

(iii) Let V be any semi*δ –open in X. since, f is pre-semi*δ-open, f(V)is a semi*δ-open in Y. Also, since g is quasi semi*δ open, g(f(V)) is open set in Z. That is, g ∘ f(V) = g(f(V)) is open in Z and hence g ∘ f is quasi semi*δ –open function

IV QUASI SEMI*δ - CLOSED FUNCTIONS

Definition 4.1: A function f: (X, τ) → (Y, σ) is said to be **quasi semi*δ-closed** if the image of every semi*δ- closed set in X is closed in Y.

Example 4.2: Let X = Y = {a, b, c, d}, τ^c = {X, ∅ , {b},{b, d},{b, c},{a, b, d},{b, c, d}} and σ^C = {Y, ∅, {a}, {a, d}, {a, c}, {a, b, d}, {a, c, d}}. In this space S*δC(X, τ) = {X,∅,{b},{c},{b, c},{a, d},{a, b, d}} and S*δC (Y, σ) = {Y, ∅,{a},{c},{a, c},{b, d}, {a, b, d}}. Let $f{:}\,(X, \tau) \to (Y, \sigma)$ be a map defined by f(a) = f(b)= f(c) =a, f(d) =c. Clearly, f is quasi semi*δ–closed.

Remark 4.3: Any semi*δ-closed functions need not be quasi semi* δ-closed as shown by following example.

Example 4.4: Let X = Y = {a, b, c, d}, τ^c = {X, ∅, {c}, {c, d}, {b, c}, {a, c, d}, {b, c, d}} and σ^c = {Y,∅,{d},{c, d},{a, d},{a, c, d},{b, c, d}} .In this space, S*δ C(X, τ) = {X,∅,{b},{c},{b, c}, {a, d},{a, c, d}} and S*δ C(Y,σ) = {X, ∅, {a}, {d}, {a, d}, {b, c}, {b, c, d}}. Let $f{:}\,(X, \tau)$→(Y,σ) be a map defined by f(a) = f(c)=a, f(b)=f(d) =d. Clearly, f is S* δ- closed. But for the semi* δ- closed set {c} in X. The image f({c})={a} is not closed in Y. Therefore f is not quasi semi* δ-closed.

Theorem 4.5: If f: (X, τ) → (Y, σ) is quasi semi*δ-closed, then $f^{-1}\left(\mathrm{int}(B)\right) \subset s^{*}\delta\mathrm{Int}\left(f^{-1}(B)\right)$for every subset B of Y.

Proof: This proof is similar to proof of lemma [3.4].

Theorem 4.6: A function $f:(X,\tau)\to(Y,\sigma)$is quasi semi*δ-closed function iff for any subset B of Y and for semi* δ-open set G of X containing $f^{-1}(B)$, there exists a open set U of Y containing B such that $f^{-1}(U)\subset G$.

Proof: This proof is similar to the theorem [3.7]

Theorem 4.7: Let $f:X\to Y$ and $g:Y\to Z$ be any two functions. Then the following properties hold:

(i) If f is semi*δ- closed and g is quasi semi*δ –closed function, then $g\circ f$ is closed function.

(ii) If f is quasi semi*δ- closed and g is semi*δ- closed, then $g\circ f$ is pre-semi*δ- closed function.

(iii) If f is pre-semi*δ- closed function and g is quasi semi*δ- closed, then $g\circ f$ is quasi semi*δ- closed.

Proof:

(i) Let B be a closed set in X. Since f is semi*δ-closed, f(B) is semi*δ-closed in Y. Since g is quasi semi*δ-closed, $g(f(B))$ is closed in Z. That is,$g\circ f(B)=g(f(B))$ is closed in Z and hence, $g\circ f$ is a closed function.

(ii) Let B be any semi*δ-closed set in X. Since f is quasi semi*δ- closed, f(B) is closed in Y. Since, g is semi*δ – closed, $g(f(B))$ is semi*δ closed in Z. That is $g\circ f(B)=g(f(B))$ is semi*δ-closed in Z. Hence $g\circ f$ is pre-semi*δ- closed function.

(iii) Let B be any semi*δ-closed set in X. Since, f is pre-semi*δ-closed, f(B) is semi*δ-closed in Y. Also since g is quasi semi*δ-closed, $g(f(B))$ is closed in Z, that is $g\circ f(B)=g(f(B))$ is closed in Z. Hence $g\circ f$ is quasi semi*δ-closed.

Theorem 4.8: Let $f:X\to Y$ and $g:Y\to Z$ be any two function such that $g\circ f$ is quasi semi*δ-closed.

(i) If f is semi*δ- irresolute surjective, then g is quasi semi*δ-closed.

(ii) If g is semi*δ- continuous injective, then f is pre-semi*δ-closed function.

Proof:

(i) Suppose that F is any semi*δ- closed set in Y. As f is semi*δ- irresolute, f^{-1} (F) is semi*δ- closed in X. Since, $g\circ f$ is quasi semi*δ- closed and f is surjective, $(g\circ f(f^{-1}(F)))=g(F)$, which is closed in Z. This implies that g is a quasi semi*δ-closed function.

(ii) Suppose F is any semi*δ-closed in X. Since, $g\circ f$ is quasi semi*δ-closed,$g\circ f(F)$ is closed in Z. Again, g is a semi*δ- continuous injective, $g^{-1}\big(g\circ f(F)\big)=f(F)$, which is semi*δ-closed in Y. This shows that f is pre-semi*δ- closed function.

REFERENCES

[1] Dunham. W., A New Closure Operator for Non-T1 Topologies, *Kyungpook Math. J. 22 (1982), 55-60*.

[2] Levine. N, Generalized closed sets in topology, Rend. Circ. Math.. Palermo, (1970), 89 – 96.

[3] Levine. N, Semi – open sets and semi – continuity in topological spaces, Amer.Math.Monthly, (1963), 36-41.

[4] Pious Missier .S and Reena.C, On Semi*δ-Open Sets in Topological Spaces, IOSR Journal of Mathematics (Sep – Oct.2016), 01-06.

[5] Pious Missier.S and Reena.C, Between δ-Closed Sets and δ-Semi-Closed Sets, International Journal of Mathematical Archive-7(11), 2016, 60-67.

[6] Pious Missier .S and Reena.C,Continuous and Irresolute Functions Via Star Generalised Closed Sets, IJMER- 7(2), Feb 2017, 58-64.

[7] Pious Missier.S and Reena.C, Contra Semi*δ-Continuous Functions in Topological Spaces, International Journal of Mathematics And its Applications Volume 5, Issue 4-A (2017), 19-30.

[8] Willard, S., General Topology, Addison – Wesley Publishing Company, Inc.(1970).