A Study on Restrained Triple Connected Two Domination Number of a Graph

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Abstract: The concept of triple connected graphs with real life application was introduced by considering the existence of a path containing any three vertices of a graph G.G. Mahadevan etc. al., introduced the concept of triple connected domination number of a graph. In this paper, we introduce a new domination parameter, called restrained triple connected two domination number of a graph. A subset S of V of a non – trivial graph G is said to be a restrained triple connected two dominating set, if S is a restrained two dominating set and the induced subgraph <S> is triple connected. The minimum cardinality taken over all restrained triple connected two dominating sets is called the restrained triple connected two domination number of G and is denoted by $\gamma_{2rtc}(G)$. Any restrained triple connected two dominating set with γ_{2rtc} vertices is called a γ2rtc- set of G. We determine this number for some standard and special graphs and obtain bounds for general graph. Its relationship with other graph theoretical parameters is also investigated.

Keywords: Triple connected graphs, restrained triple connected, restrained triple connected two domination

Subject Classification: 05C69

I. INTRODUCTION

All graphs considered here are finite, undirected without loops and multiple edges. Unless and otherwise stated, the graph G = (V, E) considered here have p = $|V|$ vertices and q = $|E|$ edges.

A subset *S* of *V* of a nontrivial graph *G* is called a *dominating set* of *G* if every vertex in $V - S$ is adjacent to at least one vertex in *S*. The *domination number* *γ(G)* of *G* is the minimum cardinality taken over all dominating sets in *G*. A subset *S* of *V* of a nontrivial graph *G* is called a *restrained dominating set* of *G* if every vertex in $V - S$ is adjacent to at least one vertex in *S* as well as another vertex in *V* - *S*. The *restrained domination number* $\gamma_r(G)$ of *G* is the minimum cardinality taken over all restrained dominating sets in *G*. A subset S of V is said to be two dominating set if every vertex in V – S is adjacent to atleast two vertices in S. The minimum cardinality taken over all two dominating sets is called the two domination number and is denoted by $\gamma_2(G)$. A subset S of V is said to be a *restrained 2-dominating set* of G if every vertex of V - S is adjacent to at least two vertices in S and every vertex of V - S is adjacent to a vertex in V - S. The minimum cardinality taken over all restrained two dominating sets is called the *restrained two domination number* and is denoted by $\gamma_{r2}(G)$.

A subset *S* of *V* of a nontrivial graph *G* is said to be triple connected dominating set, if *S* is a dominating set and the induced sub graph <*S*> is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number and is denoted by γ_{tc}. A subset *S* of *V* of a nontrivial graph *G* is said to be *restrained triple connected dominating set*, if *S* is a restrained dominating set and the induced sub graph <*S*> is triple connected. The minimum cardinality taken over all restrained triple connected dominating sets is called the *restrained triple connected domination number* and is denoted by γ_{rtc}. A subset S of V of a non – trivial graph G is said to be a restrained triple connected two dominating set, if S is a restrained two dominating set and the induced subgraph <S> is triple connected. The minimum cardinality taken over all restrained triple connected two dominating sets is called the restrained triple connected two domination number of G and is denoted by $\gamma_{2rtc}(G)$. Any restrained triple connected two dominating set with γ_{2rtc} vertices is called a γ_{2rtc}- set of G.

Theorem 1.1: A tree is triple connected iff T ≅ P_p, p ≥ 3.
Theorem 1.2: If the induced subgraph of each connected dominating set of G has more than two pendant vertices, then G does not contain a triple connected dominating set.
Theorem 1.3: For any graph G, $\left\lceil \frac{p}{\Delta + 1} \right\rceil \leq \gamma(G)$

II. RESTRAINED TRIPLE CONNECTED TWO DOMINATION NUMBER

Definition 2.1: A subset S of V of a non – trivial graph G is said to be a restrained triple connected two dominating set, if S is a restrained two dominating set and the induced subgraph <S> is triple connected. The minimum cardinality taken over all restrained triple connected two dominating sets is called the restrained triple connected two domination number of G and is denoted by $\gamma_{2rtc}(G)$. Any restrained triple connected two dominating set with γ_{2rtc} vertices is called a γ_{2rtc}- set of G.

Example 2.2: For the graph G_1 in Figure 2.1, S = {v_3, v_4, v_6, v_7} forms a γ_{2rtc}- set. Hence $\gamma_{2rtc}(G)$ =4.

G_1:

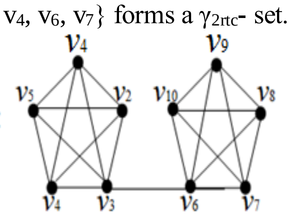

Figure 2.1

Observation 2.3: γ_{2rtc}- set does not exists for all graphs if exists $\gamma_{2rtc}(G) \geq 3$.
Observation 2.4: Every γ_{2rtc}- set is a dominating set but not conversely.
Example 2.5: For the graph G_2, in Figure 2.2 S = {v_1} is a dominating set but not a γ_{2rtc}- set.

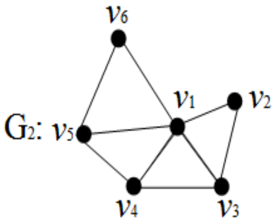

Figure 2.2

Observation 2.6: Every γ_{2rtc}- set is a connected dominating set but not conversely.
Example 2.7: For the graph G_2, in Figure 2.2 S = {v_1, v_2} is a connected dominating set but not a γ_{2rtc}- set.
Observation 2.8: Every γ_{2rtc}- set is a triple connected dominating set but not conversely
Example 2.9: For the graph G_2, in Figure 2.2 S = {v_1, v_5, v_6} is a triple connected dominating set but not a γ_{2rtc}- set.
Observation 2.10: Every γ_{2rtc}- set is a restrained triple connected dominating set but not conversely.
Example 2.11: For the graph G_2, in Figure 2.2 S = {v_1, v_5, v_6} is a triple connected dominating set but not a γ_{2rtc}- set.
Observation 2.12: The complement of the γ_{2rtc}- set need not be a γ_{2rtc}- set.
Example 2.13: For the graph G_2, in Figure 2.2 S = {v_1, v_2, v_5, v_6} is a triple connected dominating set but the complement V-S = {v_3, v_4} is not a γ_{2rtc}- set.
Theorem 2.14: For any connected graph G, $\gamma_c(G) \leq \gamma_{tc}(G) \leq \gamma_{2rtc}(G)$.

2.15 Exact value for some standard graphs:

i. For any path of order $p \geq 3$, $\gamma_{2rtc}(P_p) = p$
ii. For any cycle of order $p \geq 3$, $\gamma_{2rtc}(C_p) = p$
iii. For the complete graph of order $p \geq 3$, $\gamma_{2rtc}(K_p) = \begin{cases} p, & p = 3,4 \\ 3, & p \geq 5 \end{cases}$
iv. For the complete bipartite graph $K_{m,n}$,

$$\gamma_{2rtc}(K_{m,n}) = \begin{cases} m + n, & m \text{ or } n = 2 \\ & m \text{ or } n \geq 2 \\ 4, & m \text{ or } n \geq 3 \text{ and} \\ & m \text{ or } n \geq 3 \end{cases}$$

Theorem 2.16: If the induced subgraph of each connected dominating set of G has more than two pendant vertices, then G does not contain a restrained triple connected two dominating set.

The proof follows from Theorem 1.2

Theorem 2.17: For any connected graph G with $p \geq 3$ we have $3 \leq \gamma_{2rtc} \leq p$ and the bounds are sharp.
Proof: The lower bound follows from the definition of restrained triple connected two dominating set and the upper bound is obvious. For K_p the equality of the lower bound is attained and for C_p and P_p the equality of the upper bound is attained.
Theorem 2.18: For any connected graph G with five vertices $\gamma_{2rtc}(G) = p - 2$ iff G is isomorphic to any of the following graphs, K_5, K_4 (1), K_4 (2), K_4 (3), C_4 (3), C_4 (4), $K_4 - e$ (3).
Proof: If G is isomorphic to K_5, K_4 (1), K_4 (2), K_4 (3), C_4 (3), C_4 (4) and $K_4 - e$ (3) then it can be verified that $\gamma_{2rtc}(G) = p - 2$. Conversely let G be a connected graph with five vertices and $\gamma_{2rtc}(G) = 3$. Let S = {v_1, v_2, v_3} be the γ_{2rtc} – set of G. Take V – S = {v_4, v_5} and hence $\langle V - S \rangle = K_2$. Also $\langle S \rangle = P_3$ or C_3.

Case (i) $\langle S \rangle = P_3$ and $\langle V - S \rangle = K_2$
Let v_1, v_2, v_3 be the vertices of P_3 and v_4, v_5 be the vertices of K_2. Since G is connected v_1 (or equivalently v_3) is adjacent to v_4 (or) v_2 is adjacent to v_4 (or equivalently v_5). If v_1 is adjacent to v_4 and $d(v_4) = 3$ then we can find new graphs by increasing the degrees of v_5. If $d(v_4) = 3$, then v_4 is adjacent to (v_1 and v_2) or (v_1 and v_3) or (v_2 and v_3). If v_4 is adjacent to v_1 and v_2 then we can find new graphs by increasing the degrees of v_5 and we observe that G is isomorphic to K_4 (1), K_4 (2), C_4 (3), C_4 (4). If $d(v_4) = 4$, then v_4 is adjacent to v_1,v_2 and v_3. We can find new graphs by increasing the degrees of v_5 and we observe that G is isomorphic to K_4 (2), $K_4 - e$ (3), K_4 (3). Suppose v_2 is adjacent to v_4 then we can find new graphs by increasing the degrees of v_5. We observe that G is isomorphic to K_4 (1), C_4 (3), K_4 (2), K_4 (1) and K_4 (2), $K_4 - e$ (3).and K_4 (3).

Case (ii) $\langle S \rangle = C_3$ and $\langle V - S \rangle = K_2$.
Let v_1, v_2, v_3 be the vertices of C_3 and v_4, v_5 be the vertices of K_2. Since G is connected v_4 or v_5 is adjacent to C_3. If $d(v_4) = 3$, then v_4 is adjacent to (v_1 and v_2) or (v_1 and v_3) or (v_2 and v_3). If v_4 is adjacent to v_1 and v_2 then we can find new graphs by increasing the degrees of v_5. We observe that G is isomorphic to K_4 (2), C_4 (4) or $K_4 - e$ (3) and K_4 (3), $K_4 - e$ (3), K_4 (2) and K_4 (3). If $d(v_4) = 4$, then v_4 is adjacent to v_1,v_2 and v_3. By increasing the degree of v_5 G is isomorphic to K_4 (3) and K_5.
Theorem 2.19: Let G be a graph such that G and $\bar{G}$ have no isolates of order $p \geq 3$.Then

i. $\gamma_{2rtc}(G) + \gamma_{2rtc}(\bar{G}) \leq 2p$
ii. $\gamma_{2rtc}(G) .\gamma_{2rtc}(\bar{G}) \leq p^2$ and the bound is sharp

Proof: The bound directly follows from Theorem 2.17. For cycle C_p and path P_p equality of both the bounds are attained.
Relation with other graph parameters:

Theorem 2.20: For any connected graph G, γ_{2rtc} (G) + χ (G) $\leq$ 2p and the inequality holds if and only if G is isomorphic to K_3 or K_4 or C_3.

Proof: It is clear that, γ_{2rtc} (G) $\leq$ p and χ (G) $\leq$ p. Thus, γ_{2rtc} (G) + χ (G) $\leq$ p + p = 2p. Suppose G is isomorphic to K_3 or K_4. Then clearly γ_{2rtc}(G) + χ(G) = 2p. Conversely, let γ_{2rtc} (G) + χ (G) = 2p, the only possible case is γ_{2rtc} (G) = p and χ (G) = p. If χ (G) = p then G is isomorphic to K_p. In K_p, γ_{2rtc} (G) = 3, p $\neq$ 4 and $\gamma_{2rtc}(K_4)$ = 4 , so that G is isomorphic to K_3 or K_4. Also if G is isomorphic to C_3, then γ_{2rtc} (G) = p and χ (G) = p is possible.

Theorem 2.21: For any connected graph G γ_{2rtc}(G) + κ(G) $\leq$ 2p – 1 and the equality holds if and only if G is isomorphic to K_3 or K_4.

Proof: It is clear that γ_{2rtc}(G) $\leq$ p and κ(G) $\leq$ p – 1. Thus, γ_{2rtc}(G) + κ(G) $\leq$ p + p – 1 = 2p – 1. Suppose G is isomorphic to K_3 or K_4. Then clearly γ_{2rtc}(G) + κ(G) = 2p – 1. Conversely, let γ_{2rtc}(G) + κ(G) = 2p – 1, then the only possible case is γ_{2rtc}(G) = p and κ(G) = p – 1. Since κ(G) = p – 1, G is a complete graph. In K_p, γ_{2rtc}(G) = 3, p $\neq$ 4 and $\gamma_{2rtc}(K_4)$ = 4. Hence G is isomorphic to K_3 or K_4.

Theorem 2.22: For any connected graph G with p $\geq$ 3 vertices, γ_{2rtc}(G) + Δ (G) $\leq$ 2p – 1 and the bound is sharp.

Proof: Let G be a connected graph with p $\geq$ 3 vertices. We know that, Δ(G) $\leq$ p – 1 and by Theorem 2.17 γ_{2rtc}(G) $\leq$ p. Hence γ_{2rtc}(G) + Δ (G) $\leq$ 2p – 1 . For K_3 and K_4 the bound is sharp.

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