NUMERICAL SOLUTION OF TIME FRACTIONAL TELEGRAPH FITZHUGH-NAGUMO EQUATION

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Abstract: In this paper, we develop the fractional order Adomian Decomposition Method for the time fractional Telegraph FitzHugh-Nagumo equation. We study combination of two wave equations, Telegraph equation and FitzHugh- Nagumo equation. Time fractional derivatives are considered as Caputo fractional derivatives. Example is illustrated to show the efficiency and convergence of Adomian Decomposition Method. Also, the solutions of illustrative example is represented graphically with the help of Mathematica.

IndexTerms: Time fractional Telegraph FitzHugh-Nagumo partial differential equation, Caputo fractional derivative, Adomian Decomposition Method, Mathematica

I. INTRODUCTION

Recently, ordinary and partial differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [3], [5], [7], [8], [9], [13]. Also, many phenomenons in engineering, physics and other science can be described very successfully by models using mathematical tools from fractional calculus. Fractional derivatives provide more accurate models of real world problems than integer order derivatives. Because of their many applications in scientific fields, fractional partial differential equations are found to be an effective tool to describe certain physical phenomenon, such as diffusion processes [10], electrical and rheological materials properties [4]. Diffusion is described as spreading out one substance through another due to molecular motion rather than flow. The work of the German physiologist Adolf Eugen Fick. Engineering problems involve transmission of signals in transmission media. Many times signals are decayed due to resistance from media. This loss of signals can be determined by formation of partial differential equations. Telegraph equation is formulated by study of wave propagation. It was developed by Oliver Heaviside who worked on transmission line model. The Telegraph equation is useful in many fields such as wave propagation, random walk thery, signal analysis etc. FitzHugh - Nagumo model is a reaction-diffusion equation which also describes wave propagation in a medium. A.A.Soliman has solved this equation in the paper, "Numerical Simulation of FitzHugh-Nagumo equation". Time fractional FitzHugh-Nagumo equation was analyzed by Fairouz Tchier et.al. with Residual Power Series Method. Modified Trial Equation Method is used to find the exact solution of the time fractional FitzHugh-Nagumo equation by Yusuf Pandir. We observe that researchers have solved these two equations separately by various methods. In this paper we will solve and compare their solution by using Adomian Decomposition Method .The Adomian Decomposition Method is useful to solve effectively and accurately a large class of linear, nonlinear initial value problems and initial boundary value problems. We organize this paper as follows: In section 2, we define some basic preliminaries and properties of fractional calculus. Section 3, is developed for detailed analysis of time fractional Adomian Decomposition Method for Telegraph and FitzHugh-Nagumo equation. In the last section, we present a example to show the applicability and efficiency of the method and also the solution is demonstrated with the help of Mathematica.

II. BASIC PRELIMINARIES AND PROPERTIES OF FRACTION

In this section, we study some definitions and properties of fractional calculus.

Definition 2.1: *A real function* $f(x)$, $x > 0$, *is said to be in the space* C_μ $\mu \in R$ *if there exists a real number* $p > \mu$, *such that* $f(x) = x^p f_1(x)$, *where* $f_1(x) \in C[0,\infty)$ *and it is said to be in the space* C_μ^m *if and only if* $f^{(m)}(x) \in C\mu$, $m \in N$.

Definition 2.2 : *The Riemann - Liouville fractional integral operator of order* $\alpha \geq 0$, *of a function* $f \in C\mu$,, $\mu \geq -1$, *is defined as [11], [12]*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt \text{ , } \alpha > 0, x > 0$$

$$J^0 f(x) = f(x)$$

Properties:

For $f(x) \in C\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$, we have

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

(iii) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Definition 2.3 The Caputo derivative of fractional order α of a function $f(t)$, $f(t) \in C_{-1}^m$ is defined as follows

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(t-\tau)^{(1-m+\alpha)}}\, d\tau, \text{ for } m\text{-}1 < \alpha \leq m, m \in N, x > 0,$$

Lemma 2.1*: If* $m\text{-}1 < \alpha \leq m$ *and* $f \in C_\mu^m$, $\mu \geq -1$, *then*

$$D_*^\alpha J^\alpha f(x) = f(x)$$

$$J^{\alpha} D_{*}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^k}{k!} f^{(k)}(0+), \; x>0$$

In the next section, we develop the fractional Adomain decomposition method for fractional partial differential equation.

III. THE FRACTIONAL ADOMIAN DECOMPOSITION METHOD (FADM)

In order to elucidate the solution procedure of the FADM, we consider the following general fractional partial differential equation.

$$L^{\alpha} u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \; m-1 \leq \alpha \leq m, \; x > 0, \; t > 0 \tag{3.1}$$

Where L is fractional order derivative, R is linear differential operator, N is nonlinear operator and g is source term. Let

$$L^{\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}} \tag{3.2}$$

is the $(n\alpha)th$ order fractional derivative then the corresponding $L^{-\alpha}$ operator will be written in the following form

$$J^{\alpha} = L^{-\alpha} = \frac{1}{\Gamma^{n}(\alpha+1)} \int_0^t \int_0^{\tau_n} \int_0^{\tau_{n-1}} \ldots \int_0^{\tau_2} (d\tau_1)^{\alpha} (d\tau_2)^{\alpha} (d\tau_3)^{\alpha} \ldots (d\tau_n)^{\alpha} \tag{3.3}$$

and

$\frac{1}{\Gamma(\alpha+1)} \int_0^t (d\tau_n)^{\alpha}$ is the Caputo integration.

Operating with the operator J^{α} on both sides of equation (3.1) we have

$$J^{\alpha}[L^{\alpha} u(x, t) + Ru(x, t) + Nu(x, t)] = J^{\alpha} g(x, t)$$

$$J^{\alpha} L^{\alpha} u(x, t) = -J^{\alpha} [Ru(x, t) + Nu(x, t)] + J^{\alpha} g(x, t) \tag{3.4}$$

Using lemma (2.1) in equation (3.4), we get

$$u(x, t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \frac{\partial^k u(x,0)}{\partial t^k} - J^{\alpha} [Ru(x, t) + Nu(x, t)] + J^{\alpha} g(x, t), \; m-1 \leq \alpha \leq m \tag{3.5}$$

Now, we decompose the unknown function $u(x, t)$ into sum of an infinite number of components given by the decomposition series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \tag{3.6}$$

The nonlinear terms $Nu(x, t)$ are decomposed in the following form:

$$Nu(x, t) = \sum_{n=0}^{\infty} A_n \tag{3.7}$$

where the Adomian polynomial can be determined as follows

$$An = \frac{1}{n!} \left[\frac{d^n N}{d\lambda^n} \left(\sum_{k=0}^{\infty} \lambda^k u_k \right) \right]_{\lambda=0} \tag{3.8}$$

where An is called Adomian polynomial and that can be easily calculated by Mathematica software. Substituting the decomposition series (3.6) -(3.8) into both sides of equation (3.5) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \frac{\partial^k u(x,0)}{\partial t^k} - J^{\alpha} \left[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} An \right] + J^{\alpha} g(x, t) \tag{3.9}$$

The components $u_n(x, t)$, $n \geq 0$ of the solution $u(x, t)$ can be recursively determined by using the relation as follows:

$$u_0(x,t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \frac{\partial^k u(x,0)}{\partial t^k} + J^{\alpha} g(x,t)$$

$$u_1(x,t) = J^{\alpha}(Ru_0 + A0)$$

$$u_2(x,t) = J^{\alpha}(Ru_1 + A1)$$

$$u_3(x,t) = J^{\alpha}(Ru_2 + A2)$$

$$\vdots$$

$$u_{n+1}(x,t) = J^{\alpha}(Ru_n + An)$$

where each component can be determined by using the preceding components and we can obtain the solution in a series form by calculating the components $u_n(x, t)$, $n \geq 0$. Finally, we approximate the solution $u(x, t)$ by the truncated series.

$$\phi_N(x, t) = \sum_{n=0}^{N-1} u_n(x, t)$$

$$\lim_{N\to\infty} \emptyset_N = u(x, t)$$

In the next section, we illustrate some examples and their solutions are represented graphically by Mathematica software.

IV. APPLICATIONS

FADM for Time Fractional Telegraph FitzHugh-Nagumo Partial Differential Equation:

Consider time fractional Telegraph FitzHugh-Nagumo partial differential equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} - au(x,t) + (1+a)u^2(x,t) - u^3(x,t) \tag{4.1}$$

on a finite domain $x_L < x < x_R$ with $0 < \alpha \leq 1$. The operator form of (4.1) can be written as

$L^{\alpha} u(x,t) = D_x^2 u(x,t) - D_t^2 u(x,t) - au(x,t) + (1+a)u^2(x,t) - u^3(x,t)$

Therefore by FADM we can write

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + J^{\alpha} \left[D_x^2 \sum_{n=0}^{\infty} u_n(x,t) - D_t^2 \sum_{n=0}^{\infty} u_n(x,t) - a\sum_{n=0}^{\infty} u_n + (1+a)\sum_{n=0}^{\infty} An - \sum_{n=0}^{\infty} An' \right]$$

Then each term of the series is given by the recurrence relation,

$$u_0(x,t) = u(x,0) \tag{4.2}$$

$$u_{n+1}(x,t) = J^{\alpha}[D_x^2 u(x,t) - D_t^2 u(x,t) - au(x,t) + (1+a)An - An'] \tag{4.3}$$

It is worth noting that once the zeroth component u_0 is defined, then the remaining components u_n, $n \geq 1$ can be completely determined. Therefore, the series solution is entirely determined.

Example: Consider the time fractional partial differential equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} - 0.5u(x,t) + 1.5u^2(x,t) - u^3(x,t), \quad x>0, \; 0 < \alpha \leq 1, \; t > 0$$

Initial Condition: $u(x,0) = e^{-x^2}$

Boundary condition: $u_t(x,0) = 0, u(-6,t) = u(6,t)$

Using equation (4.2) - (4.3), we get

$u_0(x, t) = u(x, 0) = e^{-x^2}$

$u_1(x,t) = J^{\alpha}[D_x^2 u_0(x,t) - D_t^2 u_0(x,t) - 0.5u_0(x,t) + 1.5A0 - A0']$

$= J^{\alpha}[D_x^2 e^{-x^2} - D_t^2 e^{-x^2} - 0.5e^{-x^2} + 1.5A0 - A0']$

$= [e^{-x^2}(4x^2 - 2.5) + 1.5e^{-2x^2} - e^{-3x^2}]\frac{t^{\alpha}}{\Gamma(\alpha+1)}$

$u_2(x,t) = J^{\alpha}[D_x^2 u_1(x,t) - D_t^2 u_1(x,t) - 0.5u_1(x,t) + 1.5A1 - A1']$

$= [e^{-x^2}(22x^2 + 4.25) + (12x - 0.75)e^{-2x^2} - e^{-3x^2}(36x^2 - 0.5)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$

$+ [e^{-x^2}(4x^2 - 2.5) + 1.5e^{-2x^2} - e^{-3x^2}]\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)} - 6xe^{-2x^2}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 6xe^{-3x^2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$

:

:

Therefore, the series solution for the equation (4.1) is given by,

u (x, t)= u₀ (x, t) + u₁(x, t) + u₂(x, t) + u₃(x, t) + ……

Substituting values of components in above equation, we get the solution as follow

$u(x, t) = e^{-x^2} + [e^{-x^2}(4x^2 - 2.5) + 1.5e^{-2x^2} - e^{-3x^2}]\frac{t^{\alpha}}{\Gamma(\alpha+1)} + [e^{-x^2}(22x^2 + 4.25) + (12x - 0.75)e^{-2x^2} - e^{-3x^2}(36x^2 - 0.5)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + [e^{-x^2}(4x^2 - 2.5) + 1.5e^{-2x^2} - e^{-3x^2}]\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)} - 6xe^{-2x^2}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 6xe^{-3x^2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$

The Graphical representation of exact solution is as

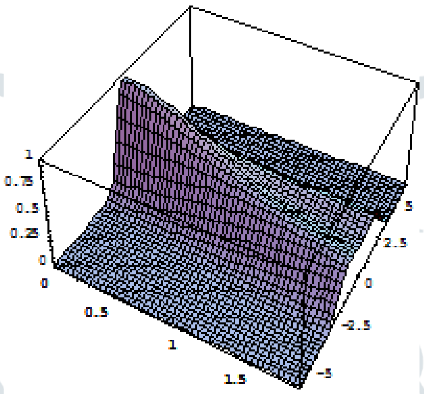

***Fig.*4.1 : The Exact solution of nonlinear partial differential equation for -6 < *x* < 6**

CONCLUSIONS

In this paper ADM has been successfully applied to find the solution of time fractional partial differential equations. ADM is very efficient and powerful technique in finding the solutions of the proposed time fractional partial differential equations. The obtained results demonstrate the reliability of the algorithm and its wider applicability to fractional partial differential equations. Finally, we come to the conclusion that the numerical results obtained by ADM are highly accurate than the numerical technique of discretization.

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