Notions via r -fuzzy $\tilde{e}$-open Sets

K. Balasubramaniyan
Department of Mathematics
Government Arts and Science College for Women
Bargur, Krishnagri, Tamil Nadu-635 104, Mathematics Section, FEAT, Annamalai University, Annamalai Nagar, Chidambaram, Tamil Nadu, India.

Abstract

In this paper the concept of r-fuzzy $\tilde{e}$-border, r-fuzzy $\tilde{e}$-exterior and r-fuzzy $\tilde{e}$-frontier in the sense of Ramadan [3] and S$\ddot{o}$stak [7] are introduced. Some interesting properties and characterizations of them are investigated. Interrelations among the concepts introduced are studied with relevant examples.

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1.Introduction

The concept of fuzzy set was introduced by Zadeh [10] in his classical paper. Fuzzy sets have applications in many fields such as information [6] and control [8]. In 1985, Sostak [7] established a new form of fuzzy topological structure. The concept of fuzzy e-open set was introduced and studied by Seenivasan [5]. The concept of fuzzy e-$T_{\frac{1}{2}}$-space was introduced and studied by [5]. The concept of g-border, g-frontier were studied in [1]. In this paper, the concepts of r-fuzzy $\tilde{e}$-border, r-fuzzy $\tilde{e}$-exterior, r-fuzzy $\tilde{e}$-frontier in the sense of Sostak [7] and Ramadam [3] are introduced.

Throughout this paper, let X be a non-empty set, $I=[0,1]$ and $I_0=(0,1]$.

2. Preliminaries

Defintion 2.1 *[4] A function $T:I^X \to I$ is called a smooth topology on X if it satisfies the following conditions:*

(i) $T(\bar{0})=T(\bar{1})=1$.

(ii) $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.

(iii) $T(\bigvee_{i\in\Gamma}\mu_i) \geq \bigwedge_{i\in\Gamma} T(\mu_i)$ for any $\{\mu_i\}_{i\in\Gamma} \in I^X$.

The pair (X,T) is called a smooth topological space

Remark 2.1 *Let (X,T) be a smooth topological space. Then, for each $r\in I_0, T_r=\{\mu\in I^X; T(\mu)\geq r\}$ is Chang's fuzzy topology on X.*

Proposition 2.1 [4] *Let* (X,T) *be a smooth topological space. For each* $\lambda \in I^X, r \in I_0$ *an operator* $C_\tau : I^X \times I_0 \to I^X$ *is defined as follows:*

$C_\tau(\lambda,r) = \bigwedge\{\mu : \mu \geq \lambda, T(\bar{1}-\mu) \geq r\}$. For $\lambda,\mu \in I^X$ and $r,s \in I_0$ it satisfies the following conditions:

(1) $C_\tau(\bar{0},r) = \bar{0}$.

(2) $\lambda \leq C_\tau(\lambda,r)$.

(3) $C_\tau(\lambda,r) \vee C_\tau(\mu,r) = C_\tau(\lambda \vee \mu,r)$.

(4) $C_\tau(\lambda,r) \leq C_\tau(\lambda,s)$ if $r \leq s$.

(5) $C_\tau(C_\tau(\lambda,r),r) = C_\tau(\lambda,r)$.

Proposition 2.2. [3] *Let* (X,T) *be a smooth topological space. For each* $\lambda \in I^X, r \in I_0$ *an operator* $I_\tau : I^X \times I_0 \to I^X$ *is defined as follows:*

$I_\tau(\lambda,r) = \bigvee\{\mu : \mu \leq \lambda, T(\mu) \geq r\}$. For each $\lambda,\mu \in I^X$ and $r,s \in I_0$ it satisfies the following conditions:

(1) $I_\tau(\bar{1}-\lambda,r) = \bar{1} - C_\tau(\lambda,r)$

(2) $I_\tau(\bar{1},r) = \bar{1}$.

(3) $I_\tau(\lambda,r) \leq \lambda$

(4) $I_\tau(\lambda,r) \wedge I_\tau(\mu,r) = I_\tau(\lambda \wedge \mu,r)$.

(5) $I_\tau(\lambda,r) \geq I_\tau(\lambda,s)$ if $r \leq s$.

(6) $I_\tau(I_\tau(\lambda,r),r) = I_\tau(\lambda,r)$.

Definition 2.2 *[2] Let* (X,τ) *be a fuzzy topological space,* $\lambda \in I^X$ *and* $r \in I_0$. *Then*

(1) A fuzzy set λ is called r-fuzzy regular open (for short, r-fro) if $\lambda = I_\tau(C_\tau(\lambda,r),r)$.

(2) A fuzzy set λ is called r-fuzzy regular closed (for short, r-frc) if $\lambda = C_\tau(I_\tau(\lambda,r),r)$.

Definition 2.3 *[2] Let* (X,τ) *be a fts. For* $\lambda,\mu \in I^X$ *and* $r \in I_0$.

(1) The r-fuzzy δ closure of λ, denoted by $\delta - C_\tau(\lambda,r)$, and is defined by $\delta - C_\tau(\lambda,r) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda, \mu$ is r-frc $\}$.

(2) The r-fuzzy δ interior of λ, denoted by $\delta - I_\tau(\lambda,r)$, and is defined by $\delta - I_\tau(\lambda,r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda, \mu$ is r-feo $\}$.

Definition 2.4 *[9] Let* (X,τ) *be a fuzzy topological space,* $\lambda \in I^X$ *and* $r \in I_0$. *Then*

(1) a fuzzy set λ is called r-fuzzy e open (for short, r-feo) if $\lambda \leq I_\tau(\delta - C_\tau(\lambda,r),r) \vee C_\tau(\delta - I_\tau(\lambda,r),r)$.

(2) A fuzzy set λ is called r-fuzzy regular closed (for short, r-frc) if $\lambda \geq I_\tau(\delta - C_\tau(\lambda,r),r) \wedge C_\tau(\delta - I_\tau(\lambda,r),r)$.

Definition 2..5 *[9] Let* (X,τ) *be a fts. For* $\lambda,\mu\in I^X$ *and* $r\in I_0$.

(1) The r-fuzzy e closure of λ, denoted by fe-$C_\tau(\lambda,r)$, and is defined by fe-$C_\tau(\lambda,r)=\wedge\{\mu\in I^X \mid \mu\geq\lambda,\mu$ is r-fec $\}$.

(2) The r-fuzzy e interior of λ, denoted by fe-$I_\tau(\lambda,r)$, and is defined by fe-$I_\tau(\lambda,r)=\vee\{\mu\in I^X \mid \mu\leq\lambda,\mu$ is r-feo $\}$.

Lemma 2.1 *[9] In a fuzzy topological space* X,

1 Any union of r-fuzzy e-open sets is a r-fuzzy e-open set.

2 Any intersection of r-fuzzy e-closed sets is a r-fuzzy e-closed set.

3. r-fuzzy $\tilde{e}$-open sets

In this section, the concept of r-f$\tilde{e}$-border, r-f$\tilde{e}$-frontier and r-f$\tilde{e}$-exterior are introduced and its properties are studied by providing necessary examples.

Definition 3.1 *Let* (X,T) *be a smooth topological space. For* $\lambda,\mu\in I^X$ *and* $r\in I_0$.

(1) λ is called r-fuzzy $\tilde{e}$-open (briefly r-f$\tilde{e}$o) if fe-$I_\tau(\lambda,r)\geq\mu$, whenever $\lambda\geq\mu$ and $\mu\in I^X$ is r-fec.

(2) λ is called r-fuzzy $\tilde{e}$-closed (briefly r-f$\tilde{e}$c) if fe-$C_\tau(\lambda,r)\leq\mu$, whenever $\lambda\leq\mu$ and $\mu\in I^X$ is r-feo.

(3) The r-fuzzy $\tilde{e}$-interior of λ, denoted by f$\tilde{e}$-$I_T(\lambda,r)$ is defined as f$\tilde{e}$-$I_T(\lambda,r)=\vee\{\mu:\mu\leq\lambda,\mu$ is r-f$\tilde{e}$o$\}$.

(4) The r-fuzzy $\tilde{e}$-closure of λ, denoted by f$\tilde{e}$-$C_T(\lambda,r)$ is defined as f$\tilde{e}$-$C_T(\lambda,r)=\wedge\{\mu:\mu\geq\lambda,\mu$ is r-f$\tilde{e}$c$\}$.

Proposition 3.1 *Let* (X,T) *be a smooth topological space. For each* $\lambda,\mu\in I^X$ *and* $r\in I_0$*, the following statements hold:*

(1) f$\tilde{e}$-$I_T(\lambda,r)$ is the largest r-f$\tilde{e}$o set such that f$\tilde{e}$-$I_T(\lambda,r)\leq\lambda$.

(2) If λ is r-f$\tilde{e}$o, then fe-$I_T(\lambda,r)=\lambda$.

(3) If λ is r-f$\tilde{e}$o, then f$\tilde{e}$-I_T(f$\tilde{e}$-$I_T(\lambda,r),r)=$f$\tilde{e}$-$I_T(\lambda,r)$.

(4) $\bar{1}-$f$\tilde{e}$-$I_T(\lambda,r)=$f$\tilde{e}$-$C_T(\bar{1}-\lambda,r)$.

(5) $\bar{1}-$f$\tilde{e}$-$C_T(\lambda,r)=$f$\tilde{e}$-$I_T(\bar{1}-\lambda,r)$.

(6) If $\lambda\leq\mu$, then f$\tilde{e}$-$I_T(\lambda,r)\leq$f$\tilde{e}$-$I_T(\mu,r)$.

(7) f$\tilde{e}$-$I_T(\lambda\vee\mu,r)\geq$f$\tilde{e}$-$I_T(\lambda,r)\vee$f$\tilde{e}$-$I_T(\mu,r)$.

(8) f$\tilde{e}$-$I_T(\lambda\wedge\mu,r)\leq$f$\tilde{e}$-$I_T(\lambda,r)\wedge$f$\tilde{e}$-$I_T(\mu,r)$.

(9) If $\lambda\leq\mu$, then f$\tilde{e}$-$C_T(\lambda,r)\leq$f$\tilde{e}$-$C_T(\mu,r)$.

Proof. Proof of (1) and (2) is trivial.

Proof of (3) follows from (2).

(4) $\tilde{e}I_T(\lambda,r)=\bar{1}-\bigvee\{\mu:\mu\le\lambda,\mu \text{ is } r-\text{f}\tilde{e}-\text{open}\}=\bigwedge\{\bar{1}-\mu:\bar{1}-\mu\ge\bar{1}-\lambda,\bar{1}-\mu \text{ is } r-\text{f}\tilde{e}-\text{closed}\}=\text{f}\tilde{e}\text{-}C_T(\bar{1}-\lambda,r)$.

(5) $C_T(\lambda,r)=\bar{1}-\bigwedge\{\mu:\mu\ge\lambda,\mu \text{ is } r-\text{f}\tilde{e}-\text{closed}\}=\bigvee\{\bar{1}-\mu:\bar{1}-\mu\le\bar{1}-\lambda,\bar{1}-\mu \text{ is } r-\text{f}\tilde{e}-\text{open}\}=\text{f}\tilde{e}\text{-}I_T(\bar{1}-\lambda,r)$.

(6) Since $\lambda\le\mu$, $I_T(\lambda,r)=\bigvee\{\theta:\theta\le\lambda,\theta \text{ is } r-\text{f}\tilde{e}-\text{open}\}\le\bigvee\{\theta:\theta\le\mu,\theta \text{ is } r-\text{f}\tilde{e}-\text{open}\}=\text{f}\tilde{e}\text{-}I_T(\mu,r)$. Hence, $\text{f}\tilde{e}\text{-}I_T(\lambda,r)\le\text{f}\tilde{e}\text{-}I_T(\mu,r)$.

(7) $I_T(\lambda\vee\mu,r)=\bigvee\{\gamma:\gamma\le\lambda\vee\mu,\gamma \text{ is } r-\text{f}\tilde{e}-\text{open}\}\ge(\bigvee\{\gamma:\gamma\ge\lambda,\gamma \text{ is } r-\text{f}\tilde{e}-\text{open}\})\vee(\bigvee\{\gamma:\gamma\le\mu,\gamma \text{ is } r-\text{f}\tilde{e}-\text{op}$ $\text{f}\tilde{e}\text{-}I_T(\lambda,r)\vee\text{f}\tilde{e}-I_T(\mu,r)$. Hence, $\text{f}\tilde{e}\text{-}I_T(\lambda\vee\mu,r)\ge\text{f}\tilde{e}\text{-}I_T(\lambda,r)\vee\text{f}\tilde{e}\text{-}I_T(\mu,r)$.

(8) $I_T(\lambda\wedge\mu,r)=\bigvee\{\gamma:\gamma\le\lambda\vee\mu,\gamma \text{ is } r-\text{f}\tilde{e}-\text{open}\}\le(\bigvee\{\gamma:\gamma\le\lambda,\gamma \text{ is } r-\text{f}\tilde{e}-\text{open}\})\wedge(\bigvee\{\gamma:\gamma\le\mu,\gamma \text{ is } r-\text{f}\tilde{e}-\text{op}$. Hence $\text{f}\tilde{e}\text{-}I_T(\lambda\wedge\mu,r)\le\text{f}\tilde{e}\text{-}I_T(\lambda,r)\wedge\text{f}\tilde{e}\text{-}I_T(\mu,r)$.

(9) Since $\lambda\le\mu$, $\text{f}\tilde{e}\text{-}C_T(\lambda,r)=\bigwedge\{\theta:\theta\ge\lambda,\theta \text{ is } r-\text{f}\tilde{e}-\text{closed}\}\le\bigwedge\{\theta:\theta\ge\mu$, θ is r -f$\tilde{e}$-closed $\}=\text{f}\tilde{e}\text{-}C_T(\mu,r)$. Hence $\text{f}\tilde{e}\text{-}C_T(\lambda,r)\le\text{f}\tilde{e}\text{-}C_T(\mu,r)$.

Definition 3.2 *Let (X,T) be a smooth topological space. For each $\lambda\in I^X$ and $r\in I_0$, the r-fuzzy e-border of λ, denoted by $fe\text{-}b_T(\lambda,r)$ is defined as $fe\text{-}b_T(\lambda,r)=\lambda-fe\text{-}I_T(\lambda,r)$.*

Definition 3.3 *Let (X,T) be a smooth topological space. For each $\lambda\in I^X$ and $r\in I_0$, the r-$f\tilde{e}$-border of λ, denoted by $f\tilde{e}\text{-}b_T(\lambda,r)$ is defined as $f\tilde{e}\text{-}b_T(\lambda,r)=\lambda-f\tilde{e}\text{-}I_T(\lambda,r)$.*

Proposition 3.2 *Let (X,T) be a smooth topological space. For each $\lambda,\mu\in I^X$ and $r\in I_0$, the following statements hold:*

(1) $\text{f}\tilde{e}\text{-}b_T(\lambda,r)\le\text{f}\tilde{e}\text{-}b_T(\lambda,r)$.

(2) If λ is r-f$\tilde{e}$-open, then $\text{f}\tilde{e}\text{-}b_T(\lambda,r)=\bar{0}$.

(3) $\text{f}\tilde{e}\text{-}I_T(\text{f}\tilde{e}\text{-}b_T(\lambda,r),r)\le\lambda$.

(4) $\text{f}\tilde{e}\text{-}b_T(\lambda,r)\le\text{f}\tilde{e}\text{-}C_T(\bar{1}-\lambda,r)$.

(5) $\text{f}\tilde{e}\text{-}b_T(\lambda\vee\mu,r)\le\text{f}\tilde{e}\text{-}b_T(\lambda,r)\vee\text{f}\tilde{e}\text{-}b_T(\mu,r)$.

(6) $\text{f}\tilde{e}\text{-}b_T(\lambda\wedge\mu,r)\ge\text{f}\tilde{e}\text{-}b_T(\lambda,r)\wedge\text{f}\tilde{e}\text{-}b_T(\mu,r)$.

(7) $\text{f}\tilde{e}\text{-}I_T(\lambda,r)\vee\text{f}\tilde{e}\text{-}b_T(\lambda,r)\ge\text{f}\tilde{e}\text{-}I_T(\lambda,r)$.

(8) $\text{f}\tilde{e}\text{-}I_T(\lambda,r)\wedge\text{f}\tilde{e}\text{-}b_T(\lambda,r)\le\text{f}\tilde{e}\text{-}I_T(\lambda,r)$.

Proof. (1) Now, $f\tilde{e}\text{-}I_T(\lambda,r) \le f\tilde{e}\text{-}I_T(\lambda,r)$. This implies that $\lambda - f\tilde{e}\text{-}I_T(\lambda,r) \ge f\tilde{e}\text{-}I_T(\lambda,r)$. Hence $f\tilde{e}\text{-}b_T(\lambda,r) \ge f\tilde{e}\text{-}b_T(\lambda,r)$.

(2) If λ is r-$f\tilde{e}$-open then $\lambda = f\tilde{e}\text{-}b_T(\lambda,r)$. Hence $f\tilde{e}\text{-}I_T(\lambda,r) = \lambda - f\tilde{e}\text{-}I_T(\lambda,r) = \bar{0}$.

(3) Now, $f\tilde{e}\text{-}I_T(f\tilde{e}\text{-}I_T(\lambda,r),r) = f\tilde{e}\text{-}I_T\text{-}(\lambda - f\tilde{e}\text{-}I_T(\lambda,r),r) \le \lambda - f\tilde{e}\text{-}I_T(\lambda,r) \le \lambda$. Thus $f\tilde{e}\text{-}I_T(f\tilde{e}\text{-}I_T(\lambda,r),r) \le \lambda$.

(4) $f\tilde{e}\text{-}b_T(\lambda,r) = \lambda - f\tilde{e}\text{-}I_T(\lambda,r) = \lambda - (\bar{1} - f\tilde{e}\text{-}C_T(\bar{1}-\lambda,r) = f\tilde{e}\text{-}C_T(\bar{1}-\lambda,r) - (\bar{1}-\lambda) \le f\tilde{e}\text{-}C_T(\bar{1}-\lambda,r)$. Hence, $f\tilde{e}\text{-}b_T(\lambda,r) \le f\tilde{e}\text{-}C_T(\bar{1}-\lambda,r)$.

(5) $f\tilde{e}\text{-}b_T(\lambda\vee\mu,r) = (\lambda\vee\mu) - f\tilde{e}\text{-}I_T(\lambda\vee\mu,r) \le (\lambda\vee\mu) - (f\tilde{e}\text{-}I_\tau(\lambda,r) \vee f\tilde{e}\text{-}I_T(\mu,r)) = (\lambda - f\tilde{e}\text{-}I_T(\lambda,r)) \vee (\mu - f\tilde{e}\text{-}I_T(\mu,r)) = f\tilde{e}\text{-}b_T(\lambda,r) \vee f\tilde{e}\text{-}b_T(\mu,r)$. Hence, $f\tilde{e}\text{-}b_T(\lambda\vee\mu,r) \le f\tilde{e}\text{-}b_T(\lambda,r) \vee f\tilde{e}\text{-}b_T(\mu,r)$.

(6) $f\tilde{e}\text{-}b_T(\lambda\wedge\mu,r) = (\lambda\wedge\mu) - f\tilde{e}\text{-}I_T(\lambda\wedge\mu,r) \ge (\lambda\wedge\mu) - (f\tilde{e}\text{-}I_T(\lambda,r) \wedge f\tilde{e}\text{-}I_T(\mu,r) = (\lambda - f\tilde{e}\text{-}I_T(\lambda,r)) \wedge (\mu - f\tilde{e}\text{-}I_T(\mu,r) = f\tilde{e}\text{-}b_T(\lambda,r) \wedge f\tilde{e}\text{-}b_T(\mu,r)$. Hence, $f\tilde{e}\text{-}b_T(\lambda\wedge\mu,r) \ge f\tilde{e}\text{-}b_T(\lambda,r) \wedge f\tilde{e}\text{-}b_T(\mu,r)$.

Proof of (7) and (8) is trivial.

Definition 3.4 *Let (X,T) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, the r-fuzzy e-frontier of λ, denoted by $fe\text{-}Fr_T(\lambda,r)$ is defined as $fe\text{-}Fr_T(\lambda,r) = fe\text{-}C_T(\lambda,r) - fe\text{-}I_T(\lambda,r)$.*

Definition 3.5 *Let (X,T) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, the r-fuzzy $\tilde{e}$-frontier of λ, denoted by $f\tilde{e}\text{-}Fr_T(\lambda,r)$ is defined as $f\tilde{e}\text{-}Fr_T(\lambda,r) = f\tilde{e}\text{-}C_T(\lambda,r) - f\tilde{e}\text{-}I_T(\lambda,r)$.*

Proposition 3.3 *Let (X,T) be a smooth topological space. For $\lambda,\mu \in I^X$ and $r \in I_0$, the following statements hold:*

(1) $f\tilde{e}\text{-}Fr_T(\lambda,r) \le f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(2) $f\tilde{e}\text{-}b_T(\lambda,r) \le f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(3) $f\tilde{e}\text{-}Fr_T(\lambda,r) = f\tilde{e}\text{-}Fr_T(\bar{1}-\lambda,r)$.

(4) $f\tilde{e}\text{-}Fr_T(f\tilde{e}\text{-}I_T(\lambda,r),r) \le f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(5) $f\tilde{e}\text{-}Fr_T(f\tilde{e}\text{-}C_T(\lambda,r),r) \le f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(6) $f\tilde{e}\text{-}I_T(\lambda,r) \ge \lambda - f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(7) $f\tilde{e}\text{-}Fr_T(\lambda\vee\mu,r) \le f\tilde{e}\text{-}Fr_T(\lambda,r) \vee f\tilde{e}\text{-}Fr_T(\mu,r)$.

(8) $f\tilde{e}\text{-}Fr_T(\lambda\wedge\mu,r) \ge f\tilde{e}\text{-}Fr_T(\lambda,r) \wedge f\tilde{e}\text{-}Fr_T(\mu,r)$.

Proof. (1) Now, $f\tilde{e}\text{-}I_T(\lambda,r)\le f\tilde{e}\text{-}I_T(\lambda,r)$. It follows that $f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r)\ge f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r)$. Hence, $f\tilde{e}\text{-}Fr_T(\lambda,r)\ge f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(2) Now, $f\tilde{e}\text{-}b_T(\lambda,r)=\lambda-f\tilde{e}\text{-}I_T(\lambda,r)\le f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r)=f\tilde{e}\text{-}Fr_T(\lambda,r)$. Hence, $f\tilde{e}\text{-}b_T(\lambda,r)\le f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(3) $f\tilde{e}\text{-}Fr_T(\lambda,r)=f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r)=(\bar{1}-f\tilde{e}\text{-}I_T(\lambda,r))-f\tilde{e}\text{-}I_T(\bar{1}-\lambda,r)=f\tilde{e}\text{-}C_T(\bar{1}-\lambda,r)-f\tilde{e}\text{-}I_T(\bar{1}-\lambda,r)=f\tilde{e}\text{-}Fr_T(\bar{1}-\lambda,r)$. Hence, $f\tilde{e}\text{-}Fr_T(\lambda,r)=f\tilde{e}\text{-}Fr_T(\bar{1}-\lambda,r)$.

(4) $f\tilde{e}\text{-}Fr_T(f\tilde{e}\text{-}I_T(\lambda,r),r)=f\tilde{e}\text{-}C_T(f\tilde{e}\text{-}I_T(\lambda,r),r)-f\tilde{e}\text{-}I_T(f\tilde{e}\text{-}I_T(\lambda,r),r)\le f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r)=f\tilde{e}\text{-}Fr_T(\lambda,r)$. Hence $f\tilde{e}\text{-}Fr_T(f\tilde{e}\text{-}I_T(\lambda,r),r)\le f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(5) $f\tilde{e}\text{-}Fr_T(f\tilde{e}\text{-}C_T(\lambda,r),r)=f\tilde{e}\text{-}C_T(f\tilde{e}\text{-}C_T(\lambda,r),r)-f\tilde{e}\text{-}I_T(f\tilde{e}\text{-}C_T(\lambda,r),r)\le f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r)=f\tilde{e}\text{-}Fr_T(\lambda,r)$. Hence $f\tilde{e}\text{-}Fr_T(f\tilde{e}\text{-}C_T(\lambda,r),r)\le f\tilde{e}\text{-}Fr_T(\lambda,r)$.

(6) $\lambda-f\tilde{e}\text{-}Fr_T(\lambda,r)=\lambda-(f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r))\le f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}C_T(\lambda,r)+f\tilde{e}\text{-}I_T(\lambda,r)=f\tilde{e}\text{-}I_T(\lambda,r)$. Hence, $\lambda-f\tilde{e}\text{-}Fr_T(\lambda,r)\le f\tilde{e}\text{-}I_T(\lambda,r)$.

(7) $f\tilde{e}\text{-}Fr_T(\lambda\vee\mu,r)=f\tilde{e}\text{-}C_T(\lambda\vee\mu,r)-f\tilde{e}\text{-}I_T(\lambda\vee\mu,r)\le(f\tilde{e}\text{-}C_T(\lambda,r)\vee f\tilde{e}\text{-}C_T(\mu,r))-(f\tilde{e}\text{-}I_T(\lambda,r)\vee f\tilde{e}\text{-}I_T(\mu,r))=(f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r))\vee(f\tilde{e}\text{-}C_T(\mu,r)-f\tilde{e}\text{-}I_T(\mu,r))=f\tilde{e}\text{-}Fr_T(\lambda,r)\vee f\tilde{e}\text{-}I_T(\mu,r)$. Hence, $f\tilde{e}\text{-}Fr_T(\lambda\vee\mu,r)\le f\tilde{e}\text{-}Fr_T(\lambda,r)\vee f\tilde{e}\text{-}Fr_T(\mu,r)$.

(8) $f\tilde{e}\text{-}Fr_T(\lambda\wedge\mu,r)=f\tilde{e}\text{-}C_T(\lambda\wedge\mu,r)-f\tilde{e}\text{-}I_T(\lambda\wedge\mu,r)\ge(f\tilde{e}\text{-}C_T(\lambda,r)\wedge f\tilde{e}\text{-}C_T(\mu,r))\text{-}(f\tilde{e}\text{-}I_T(\lambda,r)\wedge f\tilde{e}\text{-}I_T(\mu,r))=(f\tilde{e}\text{-}C_T(\lambda,r)-f\tilde{e}\text{-}I_T(\lambda,r))\wedge(f\tilde{e}\text{-}C_T(\mu,r)-f\tilde{e}\text{-}I_T(\mu,r))=f\tilde{e}\text{-}Fr_T(\lambda,r)\wedge f\tilde{e}\text{-}I_T(\mu,r)$. Hence, $f\tilde{e}\text{-}Fr_T(\lambda\wedge\mu,r)\ge f\tilde{e}\text{-}Fr_T(\lambda,r)\wedge f\tilde{e}\text{-}Fr_T(\mu,r)$.

Defintion 3.6 *Let (X,T) be a smooth topological space. For $\lambda\in I^X$ and $r\in I_0$, the r-fuzzy e-exterior of λ, denoted by $fe\text{-}Ext_T(\lambda,r)$ is defined as $fe\text{-}Ext_T(\lambda,r)=fe\text{-}I_T(\bar{1}-\lambda,r)$.*

Definition 3.7 *Let (X,T) be a smooth topological space. For $\lambda,\mu\in I^X$ and $r\in I_0$, the r-fuzzy $\tilde{e}$-exterior of λ, denoted by $f\tilde{e}\text{-}Ext_T(\lambda,r)$ is defined as $f\tilde{e}\text{-}Ext_T(\lambda,r)=f\tilde{e}\text{-}I_T(\bar{1}-\lambda,r)$.*

Proposition 3.4 *Let (X,T) be a smooth topological space. For $\lambda,\mu\in I^X$ and $r\in I_0$, the following statement*

(1) $f\tilde{e}\text{-}Ext_T(\lambda,r)\le f\tilde{e}\text{-}Ext_T(\lambda,r)$.

(2) $f\tilde{e}\text{-}Ext_T(\lambda,r)=f\tilde{e}\text{-}I_T(\bar{1}-\lambda,r)=\bar{1}-f\tilde{e}\text{-}C_T(\lambda,r)$.

(3) $f\tilde{e}\text{-}Ext_T(f\tilde{e}\text{-}Ext_T(\lambda,r),r)=f\tilde{e}\text{-}I_T(f\tilde{e}\text{-}C_T(\lambda,r))$.

(4) If $\lambda\le\mu$, then $f\tilde{e}\text{-}Ext_T(\lambda,r)\ge f\tilde{e}\text{-}Ext_T(\mu,r)$.

(5) $f\tilde{e}\text{-}Ext_T(\bar{1},r)=\bar{0}$.

(6) $f\tilde{e}\text{-}Ext_T(\bar{0},r)=\bar{1}$.

(7) $f\tilde{e}\text{-}I_T(\lambda,r)\le f\tilde{e}\text{-}Ext_T(f\tilde{e}\text{-}Ext_T(\lambda,r),r)$.

(8) $f\tilde{e}\text{-}Ext_T(\lambda \vee \mu, r) \le f\tilde{e}\text{-}Ext_T(\lambda, r) \wedge f\tilde{e}\text{-}Ext_T(\mu, r)$.

(9) $f\tilde{e}\text{-}Ext_T(\lambda \wedge \mu, r) \ge f\tilde{e}\text{-}Ext_T(\lambda, r) \vee f\tilde{e}\text{-}Ext_T(\mu, r)$.

Proof. (1) Now, $f\tilde{e}\text{-}C_T(\lambda, r) \le f\tilde{e}\text{-}C_T(\lambda, r)$. This implies, $\bar{1} - f\tilde{e}\text{-}C_T(\lambda, r) \ge \bar{1} - f\tilde{e}\text{-}C_T(\lambda, r)$. Hence, $f\tilde{e}\text{-}I_T(\bar{1}-\lambda, r) \ge f\tilde{e}\text{-}I_T(\bar{1}-\lambda, r)$. Therefore, $f\tilde{e}\text{-}Ext_T(\lambda, r) \ge Ext_T(\lambda, r)$.

(2) Proof of (2) is trivial.

(3) $f\tilde{e}\text{-}Ext_T(f\tilde{e}\text{-}Ext_T(\lambda, r), r) = f\tilde{e}\text{-}Ext_T(f\tilde{e}\text{-}I_T(\bar{1}-\lambda, r), r) = f\tilde{e}\text{-}Ext_T(\bar{1} - f\tilde{e}\text{-}C_T(\lambda, r), r) = f\tilde{e}\text{-}I_T(\bar{1}-(\bar{1} - f\tilde{e}\text{-}C_T(\lambda, r), r)) = f\tilde{e}\text{-}I_T(\bar{1}-\bar{1} + f\tilde{e}\text{-}C_T(\lambda, r), r) = f\tilde{e}\text{-}I_T(f\tilde{e}\text{-}C_T(\lambda, r), r)$.

(4) If $\lambda \le \mu$, then $f\tilde{e}\text{-}C_T(\lambda, r) \le f\tilde{e}\text{-}C_T(\mu, r)$. This implies $\bar{1} - f\tilde{e}\text{-}C_T(\lambda, r) \ge \bar{1} - f\tilde{e}\text{-}C_T(\mu, r)$. Therefore, $f\tilde{e}\text{-}I_T(\bar{1}-\lambda, r) \ge f\tilde{e}\text{-}I_T(\bar{1}-\mu, r)$. Hence, $f\tilde{e}\text{-}Ext_T(\lambda, r) \ge f\tilde{e}\text{-}Ext_T(\mu, r)$.

(5) $f\tilde{e}\text{-}Ext_T(\bar{1}, r) = f\tilde{e}\text{-}I_T(\bar{1}-\bar{1}, r) = f\tilde{e}\text{-}I_T(\bar{0}, r) = \bar{0}$. Since $\bar{0}$ itself is a r-$f\tilde{e}$-open set.

(6) $f\tilde{e}\text{-}Ext_T(\bar{0}, r) = f\tilde{e}\text{-}I_T(\bar{1}-\bar{0}, r) = f\tilde{e}\text{-}I_T(\bar{1}, r) = \bar{1}$. Since $\bar{1}$ itself is a r-$f\tilde{e}$-open set.

(7) $f\tilde{e}\text{-}Ext_T(f\tilde{e}\text{-}Ext_T(\lambda, r), r) = f\tilde{e}\text{-}Ext_T(\tilde{e}\text{-}I_T(\bar{1}-\lambda, r), r) = f\tilde{e}\text{-}Ext_T(\bar{1} - f\tilde{e}\text{-}C_T(\lambda, r), r) = f\tilde{e}\text{-}I_T(f\tilde{e}\text{-}C_T(\lambda, r), r) \ge f\tilde{e}\text{-}I_T(\lambda, r)$. Hence, $f\tilde{e}\text{-}I_T(\lambda, r) \le f\tilde{e}\text{-}Ext_T(f\tilde{e}\text{-}Ext_T(\lambda, r), r)$.

(8)f

$e-Ext_{T(g)}(\lambda \vee \mu, r) = e-I_{T(g)}(\bar{1}-(\lambda \vee \mu), r) = e-I_{T(g)}((\bar{1}-\lambda) \wedge (\bar{1}-\mu), r) \le e-I_{T(g)}(\bar{1}-\lambda, r) \wedge e-I_{T(g)}(\bar{1}-\mu, r) =$

(9) $f\tilde{e}\text{-}Ext_T(\lambda \wedge \mu, r) = f\tilde{e}\text{-}I_T(\bar{1}-(\lambda \wedge \mu), r) = f\tilde{e}\text{-}I_T((\bar{1}-\lambda) \vee (\bar{1}-\mu), r) \ge f\tilde{e}\text{-}I_T(\bar{1}-\lambda, r) \vee f\tilde{e}\text{-}I_T(\bar{1}-\mu, r) = f\tilde{e}\text{-}Ext_T(\lambda, r) \vee f\tilde{e}\text{-}Ext_T(\mu, r)$.

References

[1] M. Caldas, S. Jafari and T. Noiri, *Notions via g-open sets*, Kochi J. Math., **2**, (2007), 43-50.

[2] Y. C. Kim and J. W. Park, *r-fuzzy δ-closure and r-fuzzy θ-closure sets,* J. Korea Fuzzy Logic and Intelligent systems, **10**(6) (2000), 557-563.

[3] A. A. Ramadan, S.E. Abbas and Yong Chankim, *Fuzzy-irresolute mappings in smooth fuzzy topological spaces*, The Journal of Fuzzy Mathematics, **9**, (2001), 865-877

[4] S. K. Samanta and K. C. Chattaopadhyay, *Fuzzy topology*, Fuzzy Sets and Systems, **54**, (1993), 207-221.

[5] V. Seenivasan and K. Kamala, *Fuzzy e-continuity and fuzzy e-open sets,* Annals of Fuzzy Mathematics and Informatics 8(1) (2014), 141--148.

[6] P. Smets, *The degree of belief in a fuzzy event*, Inform Sci., **25**, (1981), 1-19.

[7] A. P. Sostak, *On a fuzzy topological structure*, Revid. Cric. Maten Palermo (Ser II), **11**, (1985), 89-103.

[8] M. Sugemo, *An introductory survey of fuzzy control*, Inform. Sci., **36**, (1985), 59-83.

[9] D. Sobana, V. Chandrasekar and A. vadivel, *Fuzzy e-continuity in $\hat{S}$ostak's fuzzy topological spaces*, (submitted).

[10] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**, (1965), 338-353.