GENERALIZED $(\in,\in\vee q)$-FUZZY WEAK BI-IDEALS OF Γ-NEAR-RINGS

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Abstract: In this article, a new notion of $(\in,\in\vee q)$-fuzzy weak bi-ideals of Γ-near-rings, which is a generalized concept of fuzzy weak bi-ideals of Γ near-rings is introduced. Further we characterize $(\in,\in\vee q)$-fuzzy weak bi-ideals of Γ-near ring using homomorphism.

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1.Introduction

Near-ring was introduced by Dickenson . Gamma-near-ring was introduced by Bh. Satyanarayana [14] in 1984. Bi-ideals of Γ-near-ring was introduced by Tamizh Chelvam and Meenakumari [15]. Further they extended it to fuzzy concept in [11].

Zadeh [16] introduced the concept of fuzzy sets in 1965. In 1971, Rosenfeld [13] extended the concept of fuzzy set theory to group theory and defined fuzzy group and derived some properties. After that fuzzy concepts evolved in almost all area in algebra and many researchers contributed interesting results in fuzzy group, semi-group, semi-ring, near-ring etc. As a further extension, using quasi-coincidence of a fuzzy point was initiated by Ming and Ming [9] in 1980. A new type of fuzzy subgroup, that is, the $(\varepsilon,\varepsilon\vee q)$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [2,3] using the combined notions of belongingness " and quasi-coincidence " of fuzzy points and fuzzy sets. The idea of $(\varepsilon,\varepsilon\vee q)$-fuzzy ideals are introduced in [3].

In 1991, Abou-Zaid [1] introduced the notion of fuzzy subnear-rings and ideals in near-rings. The idea of fuzzy ideals of near-rings was first proposed by kim et al. [8]. In [6], Davvaz introduced the concept of $(\varepsilon,\varepsilon\vee q)$-fuzzy ideals in a near-ring.

Fuzzy ideals in Gama-near-rings proposed by Jun et al. [7] in 1998.

Moreover, Manikantan [10] introduced the notion of fuzzy bi-ideals of near-rings and discussed the some of its properties. Cho et al. [5] introduced the concept of weak bi-ideals and applied it to near-rings. Chinnadurai and Kadalarasi [4] studied the fuzzy weak bi-ideals of near-rings.

In this paper, we discuss a notion of $(\in,\in\vee q)$-fuzzy weak bi-ideals of Γ-near-rings, which is a generalized concept of fuzzy weak bi-ideals of Γ-near-rings. We also investigate some of its properties with examples.

2.Preliminaries

Definition 2.1

A non-empty set N with two binary operations $+$ and $.$ is called a near-ring if

(1) $(N,+)$ is a group,

(2) $(N,.)$ is a semigroup,

(3) $x.(y+z)=x.y+x.z,$ for all $x,y,z\in N$.

We use word 'near-ring' to mean 'left near-ring'. We denote xy instead of $x.y$. Note that $x0=0$ and $x(-y)=-xy$ but in general $0x\neq 0$ for some $x\in N$.

Definition 2.2

An ideal I of a near-ring N is a subset of N such that

(4) $(I,+)$ is a normal subgroup of $(N,+)$,

(5) $NI\subseteq I$,

(6) $((x+i)y-xy)\in I$ for any $i\in I$ and $x,y\in N$.

Note that I is a left ideal of N if I satisfies (4) and (5), and I is a right ideal of N if I satisfies (4) and (6).

Definition 2.3

A two sided N-subgroup of a near-ring N is a subset H of N such that

(i) $(H,+)$ is a subgroup of $(N,+)$,

(ii) $NH\subset H$,

(iii) $HN\subset H$.

If H satisfies (i) and (ii) then it is called a left N-subgroup of N. If H satisfies (i) and (iii) then it is called a right N-subgroup of N.

Definition 2.4

Let N be a near-ring. Given two subsets A and B of N, the product $AB=\{ab\,|\,a\in A,b\in B\}$. Also we define another operation '$*$' on the class of subsets of N given by $A*B=\{a(a'+b)-aa'\,|\,a,a'\in A,b\in B\}$.

Definition 2.5

A subgroup B of $(N,+)$ is said to be a bi-ideal of N if $BNB\cap BN*B\subseteq B.$

Definition 2.6

A subgroup B of $(N,+)$ is said to be a weak bi-ideal of N if $BBB\subseteq B.$ Through out this paper, χ_I is the characteristic function of the subset I of N and the characteristic function of N is denoted by N, that means, $N:N\to[0,1]$ mapping every element of N to 1.

Definition 2.7

A function λ from a nonempty set N to the unit interval $[0,1]$ is called a fuzzy subset of N. Let λ be any fuzzy subset of N, for $t\in[0,1]$ the set $\lambda_t=\{x\in N\,|\,\lambda(x)\geq t\}$ is called a level subset of λ.

Definition 2.8

A fuzzy subset λ of a group G is said to be a fuzzy subgroup of G if $\forall\, x,y\in G,$

(i) $\lambda(xy)\geq\min\{\lambda(x),\lambda(y)\},$

(ii) $\lambda(x^{-1})=\lambda(x).$

Definition 2.9

A fuzzy subset λ of a group G is said to be an $(\varepsilon, \varepsilon \vee q)$-fuzzy subgroup of G if ? $x, y \in G$ and $t, r \in (0,1]$,

(i) $x_t, y_t \in \lambda \Rightarrow (xy)_{min\{t,r\}} \varepsilon \vee q\lambda$,

(ii) $x_t \in \lambda \Rightarrow x_t^{-1} \varepsilon \vee q\lambda$.

Definition 2.10

A fuzzy subset λ of a near-ring N is called an $(\varepsilon, \varepsilon \vee q)$-fuzzy two-sided N-subgroup of N if

(i) λ is an $(\varepsilon, \varepsilon \vee q)$- fuzzy subgroup of $(N,+)$,

(ii) $\lambda(xy) \geq min\{\lambda(x), 0.5\} \forall x, y \in N$,

(iii) $\lambda(xy) \geq min\{\lambda(y), 0.5\} \forall x, y \in N$,

If λ satisfies (i) and (ii), then λ is called an $(\varepsilon, \varepsilon \vee q)$-fuzzy left N-subgroup of N.

Definition 2.11

An $(\varepsilon, \varepsilon \vee q)$-fuzzy subgroup λ of N is called an $(\varepsilon, \varepsilon \vee q)$-fuzzy bi-ideal of N if for all $a \in N$,

$\lambda(a) \geq min\{((\lambda N \lambda) \cap (\lambda N * \lambda))(a), 0.5\}$ and

$\lambda(a) \geq min\{(\lambda N \lambda)(a), 0.5\}$ if N is zero symmetric.

Definition 2.12

A fuzzy subset λ is said to be an $(\varepsilon, \varepsilon \vee q)$-fuzzy subnear-ring of N if $\forall x, y \in N$.

(i) $\lambda(x+y) \geq min\{\lambda(x), \lambda(y), 0.5\}, \forall x, y \in N$.

(ii) $\lambda(-x) \geq min\{\lambda(x), 0.5\}, \forall x \in N$.

(iii) $\lambda(xy) \geq min\{\lambda(x), \lambda(y), 0.5\}, \forall x, y \in N$.

Definition 2.13

A fuzzy subset of N is an $(\varepsilon, \varepsilon \vee q)$- fuzzy ideal of N if and only if $\forall x, y, i \in N$,

(i) $\lambda(x-y) \geq min\{\lambda(x), \lambda(y), 0.5\}$,

(ii) $\lambda(y+x-y) \geq min\{\lambda(x), 0.5\}$,

(iii) $\lambda(xy) \geq min\{\lambda(x), 0.5\}$

(iv) $\lambda(y(x+i)-yx) \geq min\{\lambda(i), 0.5\}$.

Definition 2.14

A Γ-near-ring is a triple $(M, +, \Gamma)$ where

(i) $(M,+)$- is a group,

(ii) Γ- is a nonempty set of binary operators on M such that for each $\alpha \in \Gamma, (M, +, \alpha)$- is a near-ring,

(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.15

A Γ-near-ring M is said to be zero-symmetric if $x\alpha 0 = 0$ for all $x \in M$ and $\alpha \in \Gamma$.

Throughout this paper M denotes a zero-symmetric right Γ-near-ring with atleast two elements.

Definition 2.16

A subset A of a Γ-near-ring M is called a left (resp.right) ideal of M if

(i) $(A,+)$ is a normal subgroup of $(M,+),(i.e)\, x-y\in A$ for all $x,y\in A$ and $y+x-y\in A$ for $x\in A, y\in M$

(ii) $u\alpha(x+v)-u\alpha v\in A$ (resp. $x\alpha u\in A$) for all $x\in A,\alpha\in\Gamma$ and $u,v\in M$.

Definition 2.17

Let M be a Γ-near-ring. Given two subsets A and B of M, we define $A\Gamma B=\{a\alpha b \mid a\in A, b\in B$ and $\alpha\in\Gamma\}$

Definition 2.18

A subgroup B of $(M,+)$ is a bi-ideal if and only if $B\Gamma M\Gamma B\subseteq B.$

The characteristic function of M is denoted by M, that means, $M:M\to[0,1]$ mapping every element of M to 1.

Definition 2.19

Let A and B be any two fuzzy subsets of M. Then $A\cap B, A\cup B, A+B$ and $A\circ_\Gamma A$ are fuzzy subsets of M defined by:

$(A\cap B)(x)=min\{A(x),B(x)\}.$

$(A\cup B)(x)=max\{A(x),B(x).\}$

$$(A+B)(x)=\begin{cases}\sup\limits_{x=y+z}\{\min\{A(y),B(z)\}\} & \text{if x can be expressible as x=y+z,}\\ 0 & \text{otherwise.}\end{cases}$$

$$(A\circ_\Gamma B)(x)=\begin{cases}\sup\limits_{x=y\alpha z}\{\min\{A(y),B(z)\}\} & \text{if x can be expressible as } x=y\alpha z,\\ 0 & \text{otherwise.}\end{cases}$$

for $x,y,z\in M$ and $\alpha\in\Gamma$.

Definition 2.20

A fuzzy set A of a Γ-near-ring M is called a fuzzy left (resp.right) ideal of M if

(i) $A(x-y)\geq min\{A(x),A(y)\}$, for all $x,y\in M$,

(ii) $A(y+x-y)\geq A(x)$, for all $x,y\in M$,

(iii) $A(u\alpha(x+v)-u\alpha v)\geq A(x)$, (resp. $A(x\alpha u)\geq A(x)$) for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$.

Definition 2.21

A fuzzy subset of N is an $(\varepsilon,\varepsilon\vee q)$- fuzzy ideal of N if and only if $\forall\, x,y,i\in N$,

(i) $A(x-y)\geq min\{A(x),A(y),0.5\}$,

(ii) $A(y+x-y)\geq min\{A(x),0.5\}$,

(iii) $A(xy)\geq min\{A(x),0.5\}$

(iv) $A(y(x+i)-yx)\geq min\{A(i),0.5\}$.

3. $(\varepsilon,\varepsilon\vee q)$-Fuzzy weak bi-ideal of Γ-near-rings

In this section, we introduce the notion of $(\varepsilon,\varepsilon\vee q)$ fuzzy weak bi-ideal of M and discuss some of its properties.

Definition 3.1

A subgroup W of $(M,+)$ is said to be a weak bi-ideal of M if $W\Gamma W\Gamma W \subseteq W$.

Definition 3.2

A fuzzy set A of M is called an $(\varepsilon, \varepsilon \vee q)$ fuzzy weak bi-ideal of M, if

(i) $A(x-y) \geq min\{A(x), A(y), 0.5\}$ for all $x, y \in M$

(ii) $A(x\alpha y\beta z) \geq min\{A(x), A(y), A(z), 0.5\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Example 3.3

Let $N = \{0, a, b, c\}$ be near-ring with two binary operations $+$ and $\Gamma = \{\alpha, \beta\}$ is defined as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

α	0	a	b	c
0	0	a	0	a
a	0	a	0	a
b	0	a	b	c
c	0	a	b	c

β	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	a	c	b
c	0	a	b	c

Let $A: M \to [0,1]$ be a fuzzy subset defined by $A(0) = 0.5, A(a) = 0.6, A(b) = A(c) = 0.3.$ which is not a fuzzy weak bi-ideal, but it $(\varepsilon, \varepsilon \vee q)$ - fuzzy weak bi-ideal of M.

Theorem 3.4

Let A be a fuzzy subgroup of N. Then A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M if and only if $\min\{(A \circ_\Gamma A \circ_\Gamma A)(x), 0.5\} \subseteq A.$

Proof. Assume that A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M. Let $x, y, z, y_1, y_2 \in M$ and $\alpha, \beta \in \Gamma$ such that $x = y\alpha z$ and $y = y_1\beta y_2$. Then

$$\min\{(A \circ_\Gamma A \circ_\Gamma A)(x), 0.5\} = \{\sup_{x=y\alpha z} \min\{(A \circ_\Gamma A)(y), A(z)\}, 0.5\}$$

$$= \{\sup_{x=y\alpha z} \min\{\sup_{y=y_1\beta y_2} \min\{A(y_1), A(y_2)\}, A(z)\}, 0.5\}$$

$$= \{\sup_{x=y\alpha z} \sup_{y=y_1\beta y_2} \min\{\min\{A(y_1), A(y_2)\}, A(z)\}, 0.5\}$$

$$= \{\sup_{x=y_1\beta y_2\alpha z} \min\{A(y_1), A(y_2), A(z)\}, 0.5\}$$

A *is a* $(\varepsilon, \varepsilon \vee q)$ *– fuzzy weak bi – ideal of* M,

$A(y_1\beta y_2\alpha z) \geq \min\{A(y_1), A(y_2), A(z), 0.5\} \leq \sup_{x=y_1\beta y_2\alpha z} A(y_1\beta y_2\alpha z) = A(x)$. If x can not be expressed as $x = y\alpha z$, then $\min\{(A \circ_\Gamma A \circ_\Gamma A)(x), 0.5\} = 0 \leq A(x)$. In both cases $\min\{(A \circ_\Gamma A \circ_\Gamma A)(x), 0.5\} \leq A(x)$. Conversely, assume that $\min\{(A \circ_\Gamma A \circ_\Gamma A)(x), 0.5\} \leq A(x)$. For $x', x, y, z \in M$ and $\alpha, \beta, \alpha_1, \beta_1 \in \Gamma$. let x' be such that $x' = x\alpha y\beta z$. Then

$$\begin{aligned} A(x\alpha y\beta z) = A(x') &\geq \min\{(A\circ_\Gamma A\circ_\Gamma A)(x), 0.5\} \\ &= \{\sup_{x'=(x\alpha y)\beta z} \min\{(A\circ_\Gamma A)(x\alpha y), A(z)\}, 0.5\} \\ &= \{\sup_{x'=(x\alpha y)\beta z} \min\{\sup_{x=x\beta y} \min\{A(x), A(y)\}, A(z)\}, 0.5\} \\ &= \{\sup_{x'=x\beta y\alpha z} \min\{\min\{A(x), A(y)\}, A(z)\}, 0.5\} \end{aligned}$$

Hence $A(x\alpha y\beta z) \geq \min\{A(x), A(y), A(z), 0.5\}$.

Lemma 3.5

Let A and B be an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideals of M. Then the products $A\circ_\Gamma B$ and $B\circ_\Gamma A$ are also $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideals of M.

Proof. Let A and B be an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideals of M. Then

$$\begin{aligned} (A\circ_\Gamma B)(x-y) &= \sup_{x-y=a\alpha b} \min\{A(a), B(b)\} \\ &\geq \sup_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)} \min\{A(a_1-a_2), B(b_1-b_2)\} \\ &\geq \sup \min\{\min\{A(a_1), A(a_2), 0.5\}, \min\{B(b_1), B(b_2), 0.5\}, 0.5\} \\ &= \sup \min\{\min\{A(a_1), B(b_1)\}, \min\{A(a_2), B(b_2)\}, 0.5\} \\ &\geq \min\{\sup_{x=a_1\alpha_1 b_1} \min\{A(a_1), B(b_1)\}, \sup_{y=a_2\alpha_2 b_2} \min\{A(a_2), B(b_2)\}, 0.5\} \\ &= \min\{(A\circ_\Gamma B)(x), (A\circ_\Gamma B)(y), 0.5\}. \end{aligned}$$

It follows that $A\circ_\Gamma B$ is an $(\varepsilon, \varepsilon\vee q)$-fuzzy subgroup of M. Further, it is easy to prove that, therefore $A\circ_\Gamma B$ is an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideal of M. Similarly $B\circ_\Gamma A$ is an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideal of M.

Lemma 3.6

Every $(\varepsilon, \varepsilon\vee q)$-fuzzy ideal of M is an $(\varepsilon, \varepsilon\vee q)$-fuzzy bi-ideal of M.

Proof. Let A be an $(\varepsilon, \varepsilon\vee q)$-fuzzy ideal of M. Then

$$A*M\circ_\Gamma A \subseteq A\circ_\Gamma M\circ_\Gamma M \subseteq A\circ_\Gamma M \subseteq A$$

Since A is a $(\varepsilon, \varepsilon\vee q)$-fuzzy ideal of M. This implies that $A\circ_\Gamma M\circ_\Gamma A \subseteq A$. Therefore A is an $(\varepsilon, \varepsilon\vee q)$-fuzzy bi-ideals of M.

Theorem 3.7

Every $(\varepsilon, \varepsilon\vee q)$-fuzzy bi-ideal of M is an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideal of M.

Proof. Assume that A is an $(\varepsilon, \varepsilon\vee q)$-fuzzy bi-ideal of M. Then $AMA \cap AM\circ_\Gamma A \subseteq A$. We have $A\circ_\Gamma A\circ_\Gamma A \subseteq A\circ_\Gamma M\circ_\Gamma A$. This implies that $A\circ_\Gamma A\circ_\Gamma A \subseteq A\circ_\Gamma M\circ_\Gamma A \subseteq A$. Therefore A is an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideal of M.

Theorem 3.8

Every $(\varepsilon, \varepsilon\vee q)$-fuzzy ideal of M is an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideal of M.

Proof. By Lemma 3.6, every $(\varepsilon, \varepsilon\vee q)$-fuzzy ideal of M is an $(\varepsilon, \varepsilon\vee q)$-fuzzy bi-ideal of M. By Theorem 3.7, every fuzzy bi-ideal of M is an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideal of M. Therefore A is an $(\varepsilon, \varepsilon\vee q)$-fuzzy weak bi-ideal of M.

However the converse of the Theorems 3.7 and 3.8 is not true in general which is demonstrated by the following example.

Example 3.9

Let $N=\{0,a,b,c\}$ be near-ring with two binary operations $+$ and $\Gamma=\{\alpha,\beta\}$ be a nonempty set of binary operations as shown in the following tables:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

α	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

β	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	a	c	b
c	0	a	b	c

Let $A:M\to[0,1]$ be an $(\varepsilon,\varepsilon\vee q)$-fuzzy subset defined by $A(0)=0.5, A(a)=0.6, A(b)=0.4$ and $A(c)=0.6$. Then A is a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of M. But A is not a fuzzy weak bi-ideal of M, $A(a\alpha a\beta a)=A(0)=0.5\not\geq A(a)=0.6$ and not a fuzzy ideal. Since $A(a-a)=A(0)\not\geq \min\{A(a),A(a)\}=0.6$.

Theorem 3.10

Let $\{A_i|i\in\Omega\}$ be family of $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideals of a near-ring M, Then $\bigcap_{i\in\Omega}A_i$ is also a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of M, where Ω is any index set.

Proof. Let $\{A_i|i\in\Omega\}$ be a family of $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideals of M. Let $x,y,z\in M$ and $A=\bigcap_{i\in\Omega}A_i$. Then, $A(x)=\bigcap_{i\in\Omega}A_i(x)=(\inf_{i\in\Omega}A_i)(x)=\inf_{i\in\Omega}A_i(x)$.

$$\begin{aligned}
A(x-y) &= \inf_{i\in\Omega}\{A_i(x-y)\}\\
&\geq \inf_{i\in\Omega}\min\{A_i(x),A_i(y),0.5\}\\
&= \min\{\inf_{i\in\Omega}A_i(x),\inf_{i\in\Omega}A_i(y),0.5\}\\
&= \min\{\bigcap_{i\in\Omega}A_i(x),\bigcap_{i\in\Omega}A_i(y),0.5\}\\
&= \min\{(x),A(y),0.5\}.
\end{aligned}$$

And

$$\begin{aligned}
A(x\alpha y\beta z) &= \inf_{i\in\Omega}A_i(x\alpha y\beta z)\\
&\geq \inf_{i\in\Omega}\min\{A_i(x),A_i(y),A_i(z),0.5\}\\
&= \min\{\inf_{i\in\Omega}A_i(x),\inf_{i\in\Omega}A_i(y),\inf_{i\in\Omega}A_i(z),0.5\}\\
&= \min\{\bigcap_{i\in\Omega}A_i(x),\bigcap_{i\in\Omega}A_i(y),\bigcap_{i\in\Omega}A_i(z),0.5\}\\
&= \min\{A(x),A(y),A(z),0.5\}.
\end{aligned}$$

Thus $A=\bigcap_{i\in\Omega}\{A_i,0.5\}$ is an $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideals of N.

Theorem 3.11

Let A be a fuzzy subset of M. Then A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M if and only if A_t is an $(\varepsilon, \varepsilon \vee q)$ weak bi-ideal of M, for all $t \in [0, 0.5]$.

Proof. Assume that A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M. Let $t \in [0, 0.5]$ and $x, y \in A_t$ be such that $A(x) \geq t$ and $(y) \geq t$. Then $A(x-y) \geq \min\{A(x), A(y), 0.5\} \geq \min\{t, t\} = t$. Thus $x - y \in A_t$. Let $x, y, z \in A_t$ be such that $A(x) \geq t, A(y) \geq t$ and $A(z) \geq t$. This implies that $A(xyz) \geq \min\{A(x), A(y), A(z), 0.5\} \geq \min\{t, t, t\} = t$. Therefore $xyz \in A_t$. Hence A_t is a weak bi-ideal of N. Conversely, assume that A_t is a weak bi-ideal of M, for all $t \in [0, 0.5]$. Let $x, y \in M$. Suppose $A(x-y) < \min\{A(x), A(y), 0.5\}$. Choose t such that $A(x-y) < t < \min\{A(x), A(y), 0.5\}$. This implies that $A(x) > t, A(y) > t$ and $A(x-y) < t$. Then we have $x, y \in A_t$ but $x - y \notin A_t$ a contradiction. Thus $A(x-y) \geq \min\{A(x), A(y), 0.5\}$. If possible let there exist $x, y, z \in M$ such that $A(xyz) < \min\{A(x), A(y), A(z), 0.5\}$. Choose t such that $A(xyz) < t < \min\{A(x), A(y), A(z), 0.5\}$. Then $A(x) > t, A(y) > t, A(z) > t$ and $A(xyz) < t$. So $x, y, z \in A_t$ but $xyz \notin A_t$, which is a contradiction. Hence $A(xyz) \geq \min\{A(x), A(y), A(z), 0.5\}$. Therefore A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M.

Theorem 3.12

Let B be a nonempty subset of M and A be an $(\varepsilon, \varepsilon \vee q)$-fuzzy subset of M defined by

$$A(x) = \{s \text{ x B } t \text{otherwise}$$

for some $x \in N, s, t \in [0, 0.5]$ and $s > t$. Then B is a weak bi-ideal of M if and only if A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M.

Proof. Assume that B is a weak bi-ideal of M. Let $x, y \in M$. We consider four cases:

(1) $x \in B$ and $y \in B$.
(2) $x \in B$ and $y \notin B$.
(3) $x \notin B$ and $y \in B$.
(4) $x \notin B$ and $y \notin B$.

Case (1): If $x \in B$ and $y \in B$, then $A(x) = s = A(y)$. Since B is a weak bi-ideal of M, then $x - y \in B$. Thus, $A(x-y) = s = \min\{s, s, 0.5\} = \min\{A(x), A(y), 0.5\}$.

Case (2): If $x \in B$ and $y \notin B$, then $A(x) = s$ and $A(y) = t$ So, $\min\{A(x), A(y), 0.5\} = t$ Now, $A(x-y) = s$ or t according as $x - y \in B$ or $x - y \notin B$. By assumption, $s > t$, we have $A(x-y) \geq \min\{A(x), A(y), 0.5\}$. Similarly, we can prove case (3).

Case (4): If $x, y \notin B$, then $A(x) = t = A(y)$. So, $\min\{A(x), A(y), 0.5\} = t$ Next, $A(x-y) = s$ or t according as $x - y \in B$ or $x - y \notin B$. So, $A(x-y) \geq \min\{A(x), A(y), 0.5\}$.

Now let $x, y, z \in M$. Then we have eight cases as follows.

(1) $x \in B, y \in B$ and $z \in B$.
(2) $x \notin B, y \in B$ and $z \in B$.
(3) $x \in B, y \notin B$ and $z \in B$.
(4) $x \in B, y \in B$ and $z \notin B$.
(5) $x \notin B, y \notin B$ and $z \in B$.
(6) $x \in B, y \notin B$ and $z \notin B$.
(7) $x \notin B, y \in B$ and $z \notin B$.
(8) $x \notin B, y \notin B$ and $z \notin B$.

These cases can be proved by similar arguments as like the above cases. Hence,

$A(xyz) \geq \min\{A(x), A(y), A(z), 0.5\}$. Therefore A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M. Conversely, assume that A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M. Let $x, y, z \in B$ be such that $A(x) = A(y) = A(z) = s$ Since A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M, we have $A(x-y) \geq \min\{A(x), A(y), 0.5\} = s$ and $A(xyz) \geq \min\{A(x), A(y), A(z), 0.5\} = s$ So, $x-y, xyz \in B$. Hence B is a weak bi-ideal of M.

Theorem 3.13

A nonempty subset B of M is a weak bi-ideal of M if and only if the characteristic function f_B is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M.

Proof. Straightforward form Theorem 3.12.

Theorem 3.14

Let A be an $(\varepsilon, \varepsilon \vee q)$ - fuzzy weak bi-ideal of M, then the set $M_A = \{x \in M, | A(x) = \min\{A(0), 0.5\}\}$ is a weak bi-ideal of M.

Proof. Let A be an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M. Let $x, y \in M_A$. Then $A(x) = min\{A(0), 0.5\}$, $A(y) = \{A(0), 0.5\}$ and $A(x-y) \geq \min\{A(x), A(y), 0.5\} = \min\{A(0), A(0), 0.5\} = \{A(0), 0.5\}$. So $A(x-y) = \{A(0), 0.5\}$. Thus $x-y \in M_A$. For every $x, y, z \in M_A$, we have $A(xyz) \geq \min\{A(x), A(y), A(z), 0.5\} = \min\{A(0), A(0), A(0), 0.5\} = \{A(0), 0.5\}$. Thus $xyz \in M_A$. Hence M_A is an weak bi-ideal of M.

4. Homomorphism between $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideals of Γ near-rings

In this section, we characterize fuzzy weak bi-ideals using homomorphim.

Theorem 4.1

Let $f : M \to S$ be a homomorphism between Γ near-rings M and S. If A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of S, then $f^{-1}(A)$ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M.

Proof. Let A be an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of S. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ Then

$$\begin{aligned} f^{-1}(A)(x-y) &= A(f(x-y)) \\ &= A(f(x) - f(y)) \\ &\geq \min\{A(f(x)), A(f(y)), 0.5\} \\ &= \min\{f^{-1}(A(x)), f^{-1}(A(y)), 0.5\}. \end{aligned}$$

$$\begin{aligned} f^{-1}(A)(xyz) &= A(f(xyz)) \\ &= A(f(x)f(y)f(z)) \\ &\geq \min\{A(f(x)), A(f(y)), A(f(z)), 0.5\} \\ &= \min\{f^{-1}(A(x)), f^{-1}(A(y)), f^{-1}(A(z)), 0.5\}. \end{aligned}$$

Therefore $f^{-1}(A)$ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M.

We now also state the converse of Theorem **Error! Reference source not found.** by strengthening the condition on f as follows.

Theorem 4.2

Let $f: M \to S$ be an onto homomorphism of Γ-near-rings M and S. Let A be an $(\varepsilon, \varepsilon \vee q)$ fuzzy subset of S. If $f^{-1}(A)$ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M, then A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of S.

Proof. Let $x, y, z \in S$. Then $f(a) = x, f(b) = y$ and $f(c) = z$ for some $a, b, c \in M$. It follows that

$$\begin{aligned} A(x-y) &= A(f(a) - f(b)) \\ &= A(f(a-b)) \\ &= f^{-1}(A)(a-b) \\ &\geq \min\{f^{-1}(A)(a), f^{-1}(A)(b), 0.5\} \\ &= \min\{A(f(a)), A(f(b)), 0.5\} \\ &= \min\{A(x), A(y), 0.5\}. \end{aligned}$$

And

$$\begin{aligned} A(xyz) &= A(f(a)f(b)f(c)) \\ &= A(f(abc)) \\ &= f^{-1}(A)(abc) \\ &\geq \min\{f^{-1}(A)(a), f^{-1}(A)(b), f^{-1}(A)(c), 0.5\} \\ &= \min\{A(f(a)), A(f(b)), A(f(c)), 0.5\} \\ &= \min\{A(x), A(y), A(z), 0.5\}. \end{aligned}$$

Hence A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M.

Theorem 4.3

Let $f: M \to S$ be an onto near-ring homomorphism. If A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M, then $f(A)$ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of S.

Proof. Let A be an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M. Since $f(A)(x') = \sup_{f(x)=x'} A(x)$, for $x' \in S$ and hence $f(A)$ is nonempty. Let $x', y' \in S$. Then we have

$$\{x | x \in f^{-1}(x' - y')\} \supseteq \{x - y | x \in f^{-1}(x') \text{ and } y \in f^{-1}(y')\}$$

and $\{x | x \in f^{-1}(x'y')\} \supseteq \{xy | x \in f^{-1}(x') \text{ and } y \in f^{-1}(y')\}$

$$\begin{aligned} f(A)(x'-y') &= \sup_{f(z)=x'-y'} A(z) \\ &\geq \sup_{f(x)=x',f(y)=y'} A(x-y) \\ &\geq \sup_{f(x)=x',f(y)=y'} \min\{A(x),A(y),0.5\} \\ &= \min\{\sup_{f(x)=x'} A(x), \sup_{f(y)=y'} A(y),0.5\} \\ &= \min\{f(A)(x'), f(A)(y'),0.5\}. \end{aligned}$$

$$\begin{aligned} f(A)(x'y'z') &= \sup_{f(w)=x'\alpha y'\beta z'} A(w) \\ &\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} A(xyz) \\ &\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \min\{A(x),A(y),A(z),0.5\} \\ &= \min\{\sup_{f(x)=x'} A(x), \sup_{f(y)=y'} A(y), \sup_{f(z)=z'} A(z),0.5\} \\ &= \min\{f(A)(x'), f(A)(y'), f(A)(z'),0.5\}. \end{aligned}$$

Therefore $f(A)$ is an $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of S.

5. Anti-homomorphism of $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of Γ-near-rings

In this section, we characterize fuzzy weak bi-ideals of Γ-near-rings using anti-homomorphism.

Definition 5.1

Let M and S be Γ-near-rings. A map $\theta: M\to S$ is called a (Γ-near-ring) anti-homomorphism if $\theta(x+y)=\theta(y)+\theta(x)$ and $(x\alpha y)=\theta(y)\alpha\theta(x)$ for all $x,y\in M$ and $\alpha\in\Gamma$.

Theorem 5.2

Let $f: M\to S$ be an anti-homomorphism between Γ-near-rings M and S. If A is an $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of S, then $f^{-1}(A)$ is an $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of M.

Proof. Let A be an $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of S. Let $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Then

$$\begin{aligned} f^{-1}(A)(x-y) &= A(f(x-y)) \\ &= A(f(y)-f(x)) \\ &\geq \min\{A(f(y)),A(f(x)),0.5\} \\ &= \min\{A(f(x)),A(f(y)),0.5\} \\ &= \min\{f^{-1}(A(x)), f^{-1}(A(y)),0.5\}. \end{aligned}$$

$$\begin{aligned} f^{-1}(A)(x\alpha y\beta z) &= A(f(x\alpha y\beta z)) \\ &= A(f(z)\alpha f(y)\beta f(x)) \\ &\geq \min\{A(f(z)),A(f(y)),A(f(x)),0.5\} \\ &= \min\{A(f(x)),A(f(y)),A(f(z)),0.5\} \\ &= \min\{f^{-1}(A(x)), f^{-1}(A(y)), f^{-1}(A(z)),0.5\}. \end{aligned}$$

Therefore $f^{-1}(A)$ is an $(\varepsilon,\varepsilon\vee q)$- fuzzy weak bi-ideal of M.

Theorem 5.3

Let $f: M \to S$ be an onto Γ-near-rings anti-homomorphism. If A is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M, then $f(A)$ is an $(\varepsilon, \varepsilon \vee q)$- fuzzy weak bi-ideal of S.

Proof. Let A be an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of M. Since $f(A)(x') = \sup_{f(x)=x'} (A(x))$, for $x' \in S$ and hence $f(A)$ is nonempty. Let $x', y' \in S$ and $\alpha, \beta \in \Gamma$. Then we have

$\{x \mid x \in f^{-1}(x'-y')\} \supseteq \{x-y \mid x \in f^{-1}(x')$ and $y \in f^{-1}(y')\}$

and $\{x \mid x \in f^{-1}(x'y')\} \supseteq \{x\alpha y \mid x \in f^{-1}(x')$ and $y \in f^{-1}(y')\}$

$$\begin{aligned} f(A)(x'-y') &= \sup_{f(z)=x'-y'} \{A(z)\} \\ &= \sup_{f(x)=x', f(y)=y'} \{A(x-y)\} \\ &\geq \sup_{f(x)=x', f(y)=y'} \min\{A\{(x), A(y), 0.5\} \\ &= \min\{\sup_{f(x)=x'} \{A(x)\}, \sup_{f(y)=y'} A(y)\}, 0.5\}. \\ &= \min\{f(A)(x'), f(A)(y'), 0.5\}. \end{aligned}$$

$$\begin{aligned} f(A)(x'\alpha y'\beta z') &= \sup_{f(w)=x'\alpha y'\beta z'} A(w) \\ &\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} A(x\alpha y\beta z) \\ &\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \min\{A(x), A(y), A(z), 0.5\} \\ &= \min\{\sup_{f(x)=x'} A(x), \sup_{f(y)=y'} A(y), \sup_{f(z)=z'} A(z), 0.5\} \\ &= \min\{f(A)(x'), f(A)(y'), f(A)(z'), 0.5\}. \end{aligned}$$

Therefore $f(A)$ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of S.

Conclusion:

In this article, the generalized concept of fuzzy weak bi-ideal over a Γ-near rings has been established and the relation between the generalized fuzzy weak bi-ideals and generalized fuzzy bi-ideals are discussed. Further Homomorphic and (anti homomorphic) images and pre-images of the weak bi-ideal over Γ-near-ring has been studied.

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