Index theory on a plane and its application

[1]Maulik S. Joshi, [2]Piyush V. Patel,
[1]Assistant Professor, [2]Assistant Professor
[1]Science and Humanities Department, [2] Science and Humanities Department
[1]Aditya Silver Oak Institute Of Technology, Ahmedabad, Gujarat, India.

Abstract: In this review paper we give some results in geometry whose statements are easily understandable and are interesting. For this, one needs to understand the notion of the index of the curve with respect to the vector field. Working out on the theorems about the Index of the singular point of the vector field in a plane we come out the pleasant and surprising fact that the number of peaks and the number of valleys exceeds by one the number of passes on a mountain island.

Key words: Vector field, Singular point, Index of a curve, Index of singular point, Lebesgue lemma.

I. INTRODUCTION

Definition: A vector field V is a map defined on an open subset $U \subseteq \mathbb{R}^2$ which associates every point $p = (x, y) \in U$, a vector $V(p) = (a(x, y), b(x, y))$ in $\mathbb{R}^2$.

Definition: A point (x, y) a vector field vanishes (i.e $V(x, y) = (0,0)$) is called a singular point of the vector field.
In most of our results we use the piece wise smooth segments will be the line segments or arc of the circles. One then has such curves from the boundary of a bounded open region. The union of this region and the curve is closed bounded subset of $\mathbb{R}^2$, so is compact.

Definition: Index of a closed curve: (Geometric description)
When a point travels once on a closed curve C and returned to the original position, a field vector corresponding to the vector field V will also rotate continuously and come back to the original position. These revolutions may be in either direction, may involve back and forth oscillations also .Still, the total angular change will be multiple of 2π . This integer is called Index of the closed curve C with respect to the vector field V[1].
Analytically , the index I_C of the closed curve C with respect to the vector field V is the number

$$I_C = \frac{1}{2\pi}\int_C \omega = \frac{1}{2\pi}\int_C \frac{adb - bda}{a^2 + b^2}$$

Where

$$\omega = d\left(\arctan\left(\frac{b(x,y)}{a(x,y)}\right)\right) = \frac{adb - bda}{a^2 + b^2} = \frac{(ab_x - ba_x)dx - (ab_y - ba_y)dy}{a^2 + b^2}$$

Note that C does not pass through any of the singular point of the vector field V.

Lebesgue's lemma: Given any covering of compact metric space K of open sets, there is an $\varepsilon > 0$ such that any subset of K of diameter less than ε is contained in some open set in the covering.[3]

Using Lebesgue's lemma, the number I_C is an integer.
Also one has $I_C = 0$ for the closed curve C containing the region D having no singular point of the vector field V. As a consequence of the above, we can say that the Index of the closed curve C bounds the region D is nonzero, then there exist at least one singular point of the vector field in the interior of D. This provides us the definition of the Index of the singular point.[1]

Definition: The Index of the singular point (x_0, y_0) is the Index of the circle $f_\varepsilon(t) = (x_0 + \varepsilon\cos t\,, y_0 + \varepsilon\sin t)$ for any sufficiently small $\varepsilon > 0$. [1]
The index of singular point is well defined because $I_C = I_{C'}$ for any two neighbourhoods C and $C'$of the singular point.

Theorem: Let U be an open subset of $\mathbb{R}^2$ and $V \in C^\infty(U, R)$ be a vector field. Let C be a simple closed curve bounds a region D. Suppose all the singular points of V are isolated. Then they are finite in number, and the index of the curve C is the sum of the index of the indices of the singular points.
That is,

$$I_C = \sum_{i=1}^{n} I_{C_i}$$

Where $C_1, C_2 \dots C_n$ are neighbourhoods of the singular points.

Theorem: Suppose that $V: U \to \mathbb{R}^2$ is a vector field and $V(x,y) = (a(x,y), b(x,y))$. Suppose (x_0,y_0) is a singular point of V and $J_{(x_0,y_0)}V$, the Jacobian determinant of V at (x_0,y_0) , is nonzero. Then the index of (x_0,y_0)

$$I_{(x_0,y_0)} = \begin{cases} 1 & if \;\; J_{(x_0,y_0)}V > 0 \\ -1 & if \;\; J_{(x_0,y_0)}V < 0 \end{cases}$$

As an application above results we get surprising and nice result as follows.

Result: *Suppose an island rises from the sea then on such an island (under suitable assumptions)*

$$\#\, peaks - \#\, passes + \#\, valles = 1$$

Proof: We rephrase this in terms of maxima, saddle point and minima of function and apply our previous results to prove this.

Consider a function $f \in C^\infty(\mathbb{R}^2, R)$ such that $\{(x,y) \in \mathbb{R}^2; f(x,y) = 0\}$ is a level curve of this function. For simplicity it is assume that it is a circle C of radius r, we may take centre to be $(0,0)$.

Assume that f has only isolated critical points in the interior of the circle. Further assume that $f_{xx}\, f_{yy} - {f_{xy}}^2 \neq 0$ at any of these singular points. Then for the local maxima and local minima $f_{xx}\, f_{yy} - {f_{xy}}^2 > 0$ and for the saddle points $f_{xx}\, f_{yy} - {f_{xy}}^2 < 0$.

Suppose that $V(x,y) = (f_x(x,y), f_y(x,y))$ the gradient vector field on $\mathbb{R}^2$.

Note that the singular points of V are precisely the critical points of . Also by assumption f has only isolated critical points on, using the argument of the above theorem one shows that there are finitely many singular points of V, say $(x_1,y_1), (x_2,y_2), \dots \dots, (x_n,y_n)$ inside the region bounded by C.

Using previous result, we have

$$I_C = \sum_{i=1}^{n} I_{C_i}$$

Here C_i are small circles around singular points (x_i,y_i) lying inside C such that C_i do not intersect.

Note that $f_{xx}\, f_{yy} - {f_{xy}}^2$ gives the Jacobian determinant of the vector field $V(x,y) = (f_x, f_y)$ at (x,y). So for Isolated singular points, by the above theorem we have

$I_{C_i} = 1$ if C_i are small circles around singular point (x_i,y_i) for which $f_{xx}\, f_{yy} - {f_{xy}}^2 > 0$ at (x_i,y_i).

$I_{C_i} = -1$ if C_i are small circles around singular point (x_i,y_i) for which $f_{xx}\, f_{yy} - {f_{xy}}^2 < 0$ at (x_i,y_i).

This gives that

$I_{C_i} = 1$ if the point (x_i,y_i) is either a relative maximum or a relative minimum of f and

$I_{C_i} = -1$ if the point (x_i,y_i) is a saddle point of f .

Thus we have

$$I_C = \#\,(local\ maxima) - \#\,(Saddle\ points) + \#\,(local\ minima)$$

Now we want to find I_C , the index of the circle of radius r centred at (0,0). We know that $\alpha(t) = (r\cos t, r\sin t), t \in [0,2\pi]$ is the parameterization of the positively oriented circle. As the function f is constant on, the gradient vector (f_x, f_y) is perpendicular to C at each point and has the direction of the interior normal of C.

Thus we have $(x(t), y(t)) = (r\beta(t)\cos t, r\beta(t)\sin t), \;\; t \in [0,2\pi]$, where $\beta(t) < 0$ for all t.

Now

$$I_C = \frac{1}{2\pi}\int_C \frac{adb - bda}{a^2 + b^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{a(t)b'(t) - b(t)a'(t)}{a(t)^2 + b(t)^2} dt$$

Where $a(t) = r\beta(t)\cos t$ and $b(t) = r\beta(t)\sin t$.

$$\therefore I_C = \frac{1}{2\pi}\int_0^{2\pi} \frac{r\beta(t)\cos t\,[r\beta'(t)\sin t + r\beta(t)\cos t] - r\beta(t)\sin t\,[r\beta'(t)\cos t - r\beta(t)\sin t]}{r^2\beta(t)^2(cos^2 t + sin^2 t)} dt$$

$$\Rightarrow I_C = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2\beta(t)^2(cos^2 t + sin^2 t)}{r^2\beta(t)^2(cos^2 t + sin^2 t)} dt$$

$$\Rightarrow I_C = 1$$

Thus

$$\#\,(local\ maxima) - \#\,(Saddle\ points) + \#\,(local\ minima) = 1$$

If we think about an island then one sees that local maximum point of f gives the peak point of the mountain island, local minimum point of f gives the bottom point of valley and saddle point of f gives pass on the island.

This proves that

$$\#\, peaks - \#\, passes + \#\, valles = 1.$$

II. Conclusion:

[1] Using the theory of index of the vector field of singular point we proved a very interesting and pleasant fact which may be checked by even a young school boy assuming the situation.

[2] We can prove some interesting result for the close curve by defining suitable vector field.

III. References:

[1] V.I.Arnold: Ordinary Differential Equations Prentice-Hall of India, New Delhi-1998

[2] Richard Courant and Fritz John: Introduction to Calculus and Analysis, Volume-II, Springer (India) Private Limited, New Delhi, 2009

[3] William Fulton: Algebraic Topology

[4] *http://www.math.ttu.edu/drager/Classes/01Fall/reals/lebcov.pdf*

[5] *https://en.wikipedia.org/wiki/Winding_number*