Some Labeling Techniques of Shining Alice

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Abstract **:** This paper aims to focus on some labeling methods Shining Alice Graph. We investigate Shining Alice Graph with four types of labeling; Cordial, E-cordial, Prime, Vertex prime.

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I. INTRODUCTION

We begin with simple, finite, undirected graph $G=(V(G),\ E(G))$ where $V(G)$ and $E(G)$ denotes the vertex set and the edge set respectively. For all other terminology we follow Gross [4]. We provide some useful definitions for the present work.

Definition 1.1: The graph labeling is an assignment of numbers to the vertices or edges or both subject to certain condition(s). A detailed survey of various graph labeling is explained in Gallian [3].

Definition 1.2: [3] For a graph $G=(V(G),\ E(G))$, a mapping $f:V(G)\rightarrow\{0,1\}$ is called a *binary vertex labeling* of G and f(v) is called the *label* of the vertex v of G under f. For an edge $e = uv$, the induced edge labeling $f^*:E(G)\rightarrow\{0,1\}$ defined as $f^*(uv)=|f(u)-f(v)|$. Let $v_{f(0)}$ and $v_{f(1)}$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_{f(0)}$ and $e_{f(1)}$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.3: [1] A binary vertex labeling f of a graph G is called a *cordial labeling* if $|v_{f(1)}-v_{f(0)}|\le 1$ and $|e_{f(1)}-e_{f(0)}|\le 1$. A graph G is said to be *cordial* if it admits *cordial labeling*.

Definition 1.4: [6] Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $f:E(G)\rightarrow\{0,1\}$. Define f^* on $V(G)$ by $f^*=\sum\{f(uv)/uv\in E(G)\}\ (mod\ 2)$. The function f is called an E- cordial labeling of G if $|v_{f(1)}-v_{f(0)}|\le 1$ and $|e_{f(1)}-e_{f(0)}|\le 1$. A graph is called *E-cordial* if it admits *E-cordial labeling*.

Definition 1.5: [5] A *prime labeling* of a graph G is an injective function $f:V(G)\rightarrow\{1,2,...,|V|\}$ such that for every pair of adjacent vertices u and v, $gcd(f(u),f(v)) = 1$. The graph which admits *prime labeling* is called a *prime graph.*

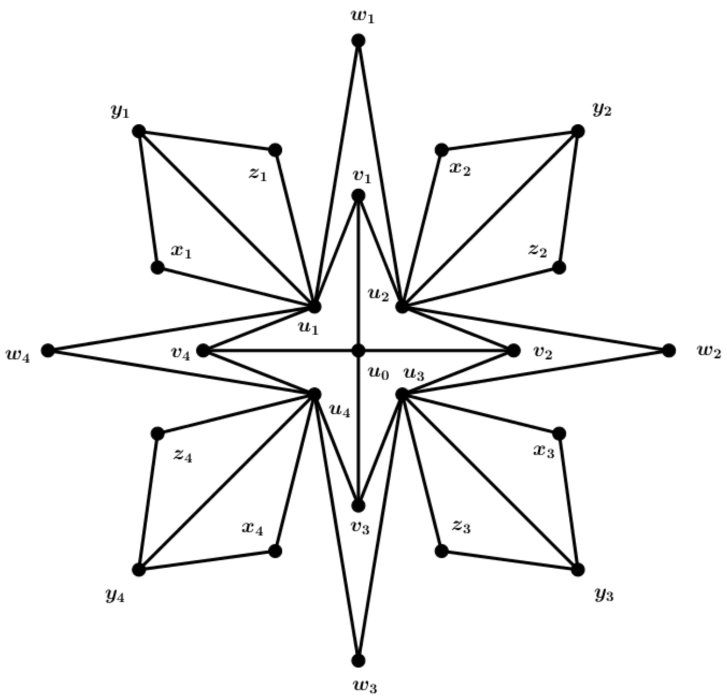
Definition 1.6: [2] For a graph G is called a *vertex prime graph* if there is a bijection $f:E(G)\rightarrow\{1,2,...,|E|\}$ such that for any vertex v, $\gcd_{uv\in E}\{f(uv)\}=1$. The bijection f is called a *vertex prime labeling* of G. The graph which admits *vertex prime labeling* is called a *vertex prime graph*.

In this paper, for every natural number n the set $\{1,2,...,n\}$ will be denoted by $[n]$.

Origami is an ancient Japanese art of folding paper. The word origami comes from two Japanese words: ``ori", which means to fold, and ``kami", which means paper. Usually origami models are made strictly by folding paper. There is no cutting or gluing involved. Even if origami is mainly an artistic product, it has received a great deal of attention from mathematicians, because of its interesting algebraic and geometrical properties. We present a new graph inspired from a model of origami namely Shining Alice.

Definition 1.7: Let $u_1,v_1,u_2,v_2,...,u_n,v_n$ be consecutive $2n$ vertices of a cycle C_{2n}; let u_0 be the apex vertex and $v_1,v_2,...,v_n$ be consecutive n pendent vertices of the star graph $K_{1,n}$; at each u_i attach a path $P_2=u_iw_iu_{i+1}$ of length 2, take n copies of fan graph $F_3=P_2\square K_1$, at each u_i attach a vertex having degree 3 of F_3, for each $i\in[n]$, subscripts are taken modulo n. The resulting graph is called ***Shining Alice*** SA_n.

Example:



II. **MAIN RESULTS:**

Theorem 2.1: SA_n is cordial.

Proof: For $SA_n, V(SA_n) = \{u_0\} \cup \{u_i, v_i, w_i, x_i, y_i, z_i \,/\, i \in [n]\}$ and $E(SA_n) = \{u_0v_i, u_iv_i, u_iw_i, u_ix_i, x_iy_i, y_iz_i, z_iu_i, u_iy_i \,/\, i \in [n]\}$ $\cup \{v_iu_{i+1}, w_iu_{i+1} \,/\, i \in [n-1]\} \cup \{v_nu_1, w_nu_1\}$. Therefor $|V(SA_n)| = 6n+1$ and $|E(SA_n)| = 10n$.

Define $f : V(SA_n) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, w_i, z_i \,/\, i \in [n]\}; \\ 0 & \text{if } x \in \{u_0\} \cup \{v_i, x_i, y_i \,/\, i \in [n]\}. \end{cases}$$

Thus $v_{f(1)} = 3n$ and $v_{f(0)} = 3n+1$. The induced edge labeling $f^* : E(SA_n) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{u_ix_i, x_iy_i, u_iy_i, u_iv_i, u_iw_i \,/\, i \in [n]\} \cup \{v_iu_{i+1} \,/\, i \in [n-1]\} \cup \{v_nu_1\}; \\ 0 & \text{if } e \in \{z_iy_i, z_iu_i, u_0v_i \,/\, i \in [n]\} \cup \{w_iu_{i+1} \,/\, i \in [n-1]\} \cup \{w_nu_1\}. \end{cases}$$

Thus $e_{f(1)} = e_{f(0)} = 5n$. Therefore f satisfies the conditions $|v_{f(1)} - v_{f(0)}| \le 1$ and $|e_{f(1)} - e_{f(0)}| \le 1$ for cordial labeling. So, f admits cordial labeling of SA_n. Hence SA_n is cordial.

Theorem 2.2: SA_n is E-cordial.

Proof: For $SA_n, V(SA_n) = \{u_0\} \cup \{u_i, v_i, w_i, x_i, y_i, z_i \,/\, i \in [n]\}$ and $E(SA_n) = \{u_0v_i, u_iv_i, u_iw_i, u_ix_i, x_iy_i, y_iz_i, z_iu_i, u_iy_i \,/\, i \in [n]\}$ $\cup \{v_iu_{i+1}, w_iu_{i+1} \,/\, i \in [n-1]\} \cup \{v_nu_1, w_nu_1\}$. Therefor $|V(SA_n)| = 6n+1$ and $|E(SA_n)| = 10n$.

Define $f : E(SA_n) \to \{0,1\}$ as follows:

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{u_ix_i, x_iy_i, u_iy_i, u_iv_i, u_iw_i \,/\, i \in [n]\}; \\ 0 & \text{if } e \in \{z_iy_i, z_iu_i, u_0v_i \,/\, i \in [n]\} \cup \{w_iu_{i+1}, v_iu_{i+1} \,/\, i \in [n-1]\} \cup \{w_nu_1, v_nu_1\}. \end{cases}$$

Thus $e_{f(1)} = e_{f(0)} = 5n$. The induced vertex labeling $f^* : V(SA_n) \to \{0,1\}$ is $f^*(V) = \sum \{f(uv) / uv \in E(SA_n)\} \ (mod\, 2)$. Therefore

$$f(x)=\begin{cases}1 & if\ \ x\in\{x_i,y_i,z_i\ /\ i\in[n]\};\\ 0 & if\ \ x\in\{u_0\}\cup\{u_i,w_i,v_i\ /\ i\in[n]\}.\end{cases}$$

Thus $v_{f(1)}=3n$ and $v_{f(0)}=3n+1$. Therefore f satisfies the conditions $|v_{f(1)}-v_{f(0)}|\le 1$ and $|e_{f(1)}-e_{f(0)}|\le 1$ for E-cordial labeling. So, f admits E-cordial labeling of SA_n. Hence SA_n is E-cordial.

Theorem 2.3: SA_n is a prime graph.

Proof: For $SA_n, V(SA_n)=\{u_0\}\cup\{u_i,v_i,w_i,x_i,y_i,z_i\ /\ i\in[n]\}$ and $E(SA_n)=\{u_0v_i,u_iv_i,u_iw_i,u_ix_i,x_iy_i,y_iz_i,z_iu_i,u_iy_i\ /\ i\in[n]\}$ $\cup\{v_iu_{i+1},w_iu_{i+1}\ /\ i\in[n-1]\}\cup\{v_nu_1,w_nu_1\}$. Therefor $|V(SA_n)|=6n+1$ and $|E(SA_n)|=10n$. Define $f:V(SA_n)\to[6n+1]$ as follows:

$$f(x)=\begin{cases}1 & if\ \ x=u_0;\\ 6i+1 & if\ \ x=u_i,\ i\in[n];\\ 6i+2 & if\ \ x=v_i,\ i\in[n-1];\\ 6i+3 & if\ \ x=w_i,\ i\in[n-1];\\ 6i-2 & if\ \ x=x_i,\ i\in[n];\\ 6i-1 & if\ \ x=y_i,\ i\in[n];\\ 6i & if\ \ x=z_i,\ i\in[n];\\ 2 & if\ \ x=v_n;\\ 3 & if\ \ x=w_n.\end{cases}$$

Clearly f is an injective function. Let e be an arbitrary edge of SA_n. To prove f is a prime labeling of SA_n we have the following:

$If\ \ e=u_0v_i,\ gcd(f(u_0),f(v_i))=gcd(1,6i+2)=1,\ i\in[n-1].$

$If\ \ e=u_0v_n,\ gcd(f(u_0),f(v_n))=gcd(1,2)=1.$

$If\ \ e=u_iv_i,\ gcd(f(u_i),f(v_i))=gcd(6i+1,6i+2)=1,\ i\in[n-1].$

$If\ \ e=u_iw_i,\ gcd(f(u_i),f(w_i))=gcd(6i+1,6i+3)=1,\ i\in[n-1].$

$If\ \ e=u_nv_n,\ gcd(f(u_n),f(v_n))=gcd(6n+1,2)=1.$

$If\ \ e=u_nw_n,\ gcd(f(u_n),f(w_n))=gcd(6n+1,3)=1.$

$If\ \ e=u_ix_i,\ gcd(f(u_i),f(x_i))=gcd(6i+1,6i-2)=1,\ i\in[n].$

$If\ \ e=u_iy_i,\ gcd(f(u_i),f(y_i))=gcd(6i+1,6i-1)=1,\ i\in[n].$

$If\ \ e=u_iz_i,\ gcd(f(u_i),f(z_i))=gcd(6i+1,6i)=1,\ i\in[n].$

$If\ \ e=x_iy_i,\ gcd(f(x_i),f(y_i))=gcd(6i-2,6i-1)=1,\ i\in[n].$

$If\ \ e=y_iz_i,\ gcd(f(y_i),f(z_i))=gcd(6i-1,6i)=1,\ i\in[n].$

$If\ \ e=v_iu_{i+1},\ gcd(f(v_i),f(u_{i+1}))=gcd(6i+2,6i+7)=1,\ i\in[n-1].$

$If\ \ e=w_iu_{i+1},\ gcd(f(w_i),f(u_{i+1}))=gcd(6i+3,6i+7)=1,\ i\in[n-1].$

$If\ \ e=v_nu_1,\ gcd(f(v_n),f(u_1))=gcd(2,7)=1.$

$If\ \ e=w_nu_1,\ gcd(f(w_n),f(u_1))=gcd(3,7)=1.$

Thus, f is an injection and $gcd(f(u),f(v))=1$ for every pair of adjacent vertices u and v of SA_n. So f admits prime labeling of SA_n. Hence SA_n is a prime graph.

Theorem 2.4: SA_n is a vertex prime graph.

Proof: For $SA_n, V(SA_n) = \{u_0\} \cup \{u_i, v_i, w_i, x_i, y_i, z_i / i \in [n]\}$ and $E(SA_n) = \{u_0v_i, u_iv_i, u_iw_i, u_ix_i, x_iy_i, y_iz_i, z_iu_i, u_iy_i / i \in [n]\}$ $\cup \{v_iu_{i+1}, w_iu_{i+1} / i \in [n-1]\} \cup \{v_nu_1, w_nu_1\}$. Therefor $|V(SA_n)| = 6n+1$ and $|E(SA_n)| = 10n$.

Define $f : E(SA_n) \to [10n]$ as follows:

$$f(x) = \begin{cases} 10i-9 & \text{if } x = u_0v_i,\ i \in [n]; \\ 10i-8 & \text{if } x = u_ix_i,\ i \in [n]; \\ 10i-7 & \text{if } x = x_iy_i,\ i \in [n]; \\ 10i-6 & \text{if } x = u_iy_i,\ i \in [n]; \\ 10i-5 & \text{if } x = y_iz_i,\ i \in [n]; \\ 10i-4 & \text{if } x = u_iz_i,\ i \in [n]; \\ 10i-3 & \text{if } x = u_iv_i,\ i \in [n]; \\ 10i-2 & \text{if } x = u_iw_i,\ i \in [n]; \\ 10i-1 & \text{if } x = w_iu_{i+1},\ i \in [n-1]; \\ 10i & \text{if } x = v_iu_{i+1},\ i \in [n-1]; \\ 10n-1 & \text{if } x = w_nu_1; \\ 10n & \text{if } x = v_nu_1. \end{cases}$$

Clearly f is a bijection. Let v be an arbitrary vertex of SA_n. To prove f is a vertex prime labeling of SA_n we have the following:

$$If\ \ v = u_0,\ gcd\left(f(u_0u_1), f(u_0u_2), \ldots, f(u_0u_n)\right) = gcd(1, 11, \ldots, 10n-9) = 1.$$

$$If\ \ v = w_i,\ gcd\left(f(u_iw_i), f(w_iu_{i+1})\right) = \gcd(10i-2, 10i-1) = 1,\ i \in [n-1].$$

$$If\ \ v = v_i,\ gcd\left(f(u_iv_i), f(v_iu_{i+1})\right) = \gcd(10i-3, 10i) = 1,\ i \in [n-1].$$

$$If\ \ v = w_n,\ gcd\left(f(u_nw_n), f(w_nu_1)\right) = \gcd(10n-2, 10n-1) = 1.$$

$$If\ \ v = v_n,\ gcd\left(f(u_nv_n), f(v_nu_1)\right) = \gcd(10n-3, 10n) = 1.$$

$$If\ \ v = x_i,\ gcd\left(f(u_ix_i), f(x_iy_i)\right) = \gcd(10i-8, 10i-7) = 1,\ i \in [n].$$

$$If\ \ v = y_i,\ gcd\left(f(x_iy_i), f(y_iz_i), f(u_iy_i)\right) = \gcd(10i-7, 10i-5, 10i-6) = 1,\ i \in [n].$$

$$If\ \ v = z_i,\ gcd\left(f(u_iz_i), f(z_iy_i)\right) = \gcd(10i-4, 10i-5) = 1,\ i \in [n].$$

$$If\ \ v = u_i,\ gcd\left(f(u_iv_i), f(u_iw_i), f(u_ix_i), f(u_iy_i), f(u_iz_i), f(u_iv_{i+1}), f(u_iw_{i+1})\right)$$
$$= gcd(10i-3, 10i-2, 10i-8, 10i-6, 10i-4, 10i+10, 10i+9) = 1, i \in [n-1].$$

$$If\ \ v = u_n,\ gcd\left(f(u_nv_n), f(u_nw_n), f(u_nx_n), f(u_ny_n), f(u_nz_n), f(u_nv_1), f(u_nw_1)\right)$$
$$= gcd(10n-3, 10n-2, 2, 4, 6, 7, 8) = 1.$$

Thus, f is a bijection and $\gcd\limits_{uv \in E}\{f(uv)\} = 1$. The edges are labeled such that for any vertex v_i, the gretest common divisor of all the edges incident with v_i is 1. So f admits vertex prime labeling of SA_n. Hence SA_n is a vertex prime graph.

III. Conclusion

We have derived four new results by investigating some labeling techniques of shining alice graph. More exploration is possible for other graph families and in the context of different graph labeling problems.

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