# Vertex Even and Odd Mean Labeling in the Context of Some Cyclic Snake Graphs

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***Abstract :*** This study has been undertaken to prove vertex even and odd mean labeling of quadrilateral and pentagonal snake graph. We have also shown alternating quadrilateral snake are vertex even and odd mean graph. It is proved that even vertex odd mean graph are even mean graph.

***IndexTerms - Vertex even mean graph, vertex odd mean graph, vertex even and odd mean graph, cyclic snake, quadrilateral snake, pentagonal snake.***

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## I. Introduction

A graph $G = (V(G); E(G))$ with $p$ vertices and $q$ edges we mean a simple graph with no loops and no parallel edges. For all graph theory terminology and notation we follow Gross and Yellen [1]. A graph labeling is an assignment of integers to the vertices or edges, or both subject to certain conditions. Many types of labeling like mean labeling, vertex odd and even mean labeling, product cordial, prime, etc. are used by various researchers in practice.

For latest survey of graph labeling we refer to Gallian [2]. Vast amount of literature is available on different types of graph labeling and more than 2000 papers have been published. We will give brief summary of definitions which are useful for present investigation.

**Definition 1.1:** A mean labeling f is an injective function from $V$ to the set $\{0, 1, 2, \dots, q\}$ such that the each edge $uv$ is assigned a label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is $\{1, 2, 3, \dots, q\}$. Somasundaram and Ponraj [3] have introduced the notion of mean labeling of graphs.

**Definition 1.2:** A graph $G$ with $q$ edges is said to be an odd mean graph if there is an injective function $f: V(G) \to \{1, 3, 5, \dots, 2q-1\}$ such that the induced function $f^*: E(G) \to N$ for each edge $uv$ assigns a label $f^*(uv) = \frac{f(u)+f(v)}{2}$, results into distinct edge labels. A function this property is called vertex odd mean labeling [5].
The notion odd mean graph was given by Manickam and Marudai [4]. Revathi investigated vertex odd and even mean labeling of umbrella graph, mongolian tent and $K_1 + C_n$ [5]. Anitha, Selvam, Thirusangu proved that the extended duplicate graph of kite graph admits mean, even mean and odd mean labeling [6].

**Definition 1.3:** A graph $G$ with $q$ edges is said to be an even mean graph if there is an injective function $f: V(G) \to \{2, 4, 6, \dots, 2q\}$ such that the induced function $f^*: E(G) \to N$ for each edge $uv$ assigns a label $f^*(uv) = \frac{f(u)+f(v)}{2}$, results into distinct edge labels. A function this property is called vertex even mean labeling.

**Definition 1.4:** A graph $G$ is said to have an even vertex odd mean labeling if there exists an injective function $f: V(G) \to \{0, 2, 4, \dots, 2q\}$ such that the induced map $f^*: E(G) \to \{1, 3, 5, \dots, 2q-1\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ is a bijection. A graph that admits even vertex odd mean labeling are called even vertex odd mean graph.
The notion of even vertex odd mean labeling is introduced by Vasuki, Nagarajan and Arockiaraj [5].

**Definition 1.5:** A cyclic snake $kC_n$ is obtain by replacing every edge of $P_k$ by $C_n$.

**Definition 1.6:** A double cyclic snake $D(kC_n)$ is obtained from two cyclic snakes that have a common path $P_k$.

**Definition 1.7:** An alternate cyclic snake $A(kC_n)$ is obtained by replacing every alternate edge of $P_k$ by $C_n$.

**Definition 1.8:** An alternate double cyclic snake $D(kC_n)$ is obtained from two alternate cyclic snakes that have a common $P_k$.

For $n = 4, 5$ we call cyclic snake as quadrilateral snake $QS_n$ and pentagonal snake $PS_n$ respectively, where $n$ denotes length of the path $P_n$.

## II. Main Results

**Theorem 2.1:** Quadrilateral snake admits vertex even and odd mean labeling.

**Proof:** Let $QS_n$ be the quadrilateral snake graph with path $P_n$. We get $|V(G)| = 3n + 1$ and $|E(G)| = 4n = q$. We label the vertices of $P_n$ with $v_1, v_4, v_7, \ldots, v_{3n+1}$. We label each upper and lower peak of each $C_4$ with $v_{3i-1}$ and $v_{3i}$, $i = 1, 2, 3, \ldots, n$ respectively. We label edges of $QS_n$ as $e_{4i-3}, e_{4i-2}, e_{4i-1}$ and $e_{4i}$ joining the vertices $v_{3i-2}v_{3i-1}$, $v_{3i-2}v_{3i}$, $v_{3i-1}v_{3i+1}$ and $v_{3i}v_{3i+1}$ respectively for $i = 1, 2, 3, \ldots, n$.

- Case 1: Define a function $f: V(G) \to \{1, 3, 5, \ldots, 2q - 1\}$ as follows:

$$f(v_i) = \begin{cases} 6k - 5, & if\ i = 3k - 2, k = 1, 2, \ldots, n + 1; \\ 6k - 3, & if\ i = 3k - 1, k = 1, 2, \ldots, n; \\ 6k - 1, & if\ i = 3k, k = 1, 2, \ldots, n. \end{cases}$$

Using above vertex labeling we obtain an induced labeling $f^*: E(G) \to N$ such that $f^*(uv) = \frac{f(u)+f(v)}{2}$ gives us:

$$f^*(e_i) = \begin{cases} 6k - 4, & \text{if } i = 4k - 3, k = 1, 2, \ldots, n; \\ 6k - 3, & \text{if } i = 4k - 2, k = 1, 2, \ldots, n; \\ 6k - 1, & \text{if } i = 4k - 1, k = 1, 2, \ldots, n; \\ 6k, & \text{if } i = 4k, k = 1, 2, \ldots, n. \end{cases}$$

Thus, all the edge labels are distinct. Hence $QS_n$ admits vertex odd mean labeling. Thus, it is vertex odd mean graph.

- Case 2: Define a function $f: V(G) \to \{2, 4, 6, \ldots, 2q\}$ as follows:

$$f(v_i) = \begin{cases} 6k - 4, & \text{if } i = 3k - 2, k = 1, 2, \ldots, n + 1; \\ 6k - 2, & \text{if } i = 3k - 1, k = 1, 2, \ldots, n; \\ 6k, & \text{if } i = 3k, k = 1, 2, \ldots, n. \end{cases}$$

Using above vertex labeling we obtain an induced labeling $f^*: E(G) \to N$ such that $f^*(uv) = \frac{f(u)+f(v)}{2}$ gives us:

$$f^*(e_i) = \begin{cases} 6k - 3, & \text{if } i = 4k - 3, k = 1, 2, \ldots, n; \\ 6k - 2, & \text{if } i = 4k - 2, k = 1, 2, \ldots, n; \\ 6k, & \text{if } i = 4k - 1, k = 1, 2, \ldots, n; \\ 6k + 1, & \text{if } i = 4k, k = 1, 2, \ldots, n. \end{cases}$$

Thus, all the edge labels are distinct. Hence $QS_n$ admits vertex even mean labeling. Thus, it is vertex even mean graph.

**Illustration 2.2:** Vertex even mean labeling of $QS_3$ is shown in figure below.

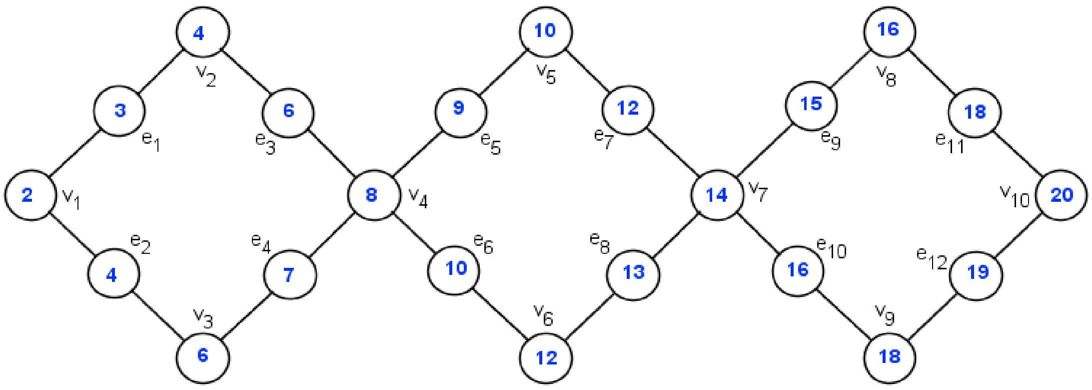


*Figure 1: Vertex even mean labeling of $QS_3$*

**Theorem 2.3:** Pentagonal snake admits vertex even and odd mean labeling.

**Proof:** Let $PS_n$ be the pentagonal snake graph with path $P_n$. We get $|V(G)| = 4n + 1$ and $|E(G)| = 5n = q$. We label the first vertex of $P_n$ with $v_1$, then we consecutively label remaining vertices with $v_{4i-2}, v_{4i-1}, v_{4i}, v_{4i+1}, for\ i = 1, 2, 3, \ldots, n$ in clockwise direction. We label the vertices which connects two cycle with $v_{4i-1}, i = 1, 2, 3, \ldots, n - 1$. We label edges of $PS_n$ as $e_1, e_2, \ldots, e_5, e_6, e_7, \ldots, e_{5n}$ in clockwise direction.

- Case 1: Define a function $f: V(G) \to \{1, 3, 5, \ldots, 2q - 1\}$ as follows:

$$f(v_i) = \begin{cases} 1, & if\ i = 1; \\ 10k - 7, & if\ i = 4k - 2, i = 1, 2, \ldots, n; \\ 10k - 5, & if\ i = 4k - 1, k = 1, 2, \ldots, n; \\ 10k - 3, & if\ i = 4k, k = 1, 2, \ldots, n; \\ 10k - 1, & if\ i = 4k + 1, k = 1, 2, \ldots, n. \end{cases}$$

Using above vertex labeling we obtain an induced labeling $f^*: E(G) \rightarrow N$ such that $f^*(uv) = \frac{f(u)+f(v)}{2}$ gives us:

$$f^*(e_i) = \begin{cases} 2, & if\ i = 1; \\ 5, & if\ i = 5; \\ 10k-1, if\ i = 5k-4, i = 2,3,\dots,n; \\ 10k-6, if\ i = 5k-3, k = 1,2,\dots,n; \\ 10k-4,, if\ i = 5k-2, k = 1,2,\dots,n; \\ 10k-2, if\ i = 5k-1, k = 1,2,\dots,n; \\ 10k-8, if\ i = 5k, k = 2,3,\dots,n. \end{cases}$$

Thus, all the edge labels are distinct. Hence $PS_n$ admits vertex odd mean labeling. Thus, it is vertex odd mean graph.

- Case 2: Define a function f: V(G) → {2, 4, 6, … , 2q} as follows:

$$f(v_i) = \begin{cases} 2, & if\ i = 1; \\ 10k-6, if\ i = 4k-2, i = 1,2,\dots,n; \\ 10k-4, if\ i = 4k-1, k = 1,2,\dots,n; \\ 10k-2, if\ i = 4k, k = 1,2,\dots,n; \\ 10k, if\ i = 4k+1, k = 1,2,\dots,n. \end{cases}$$

Using above vertex labeling we obtain an induced labeling $f^*: E(G) \rightarrow N$ such that $f^*(uv) = \frac{f(u)+f(v)}{2}$ gives us:

$$f^*(e_i) = \begin{cases} 3, & if\ i = 1; \\ 6, & if\ i = 5; \\ 10k, if\ i = 5k-4, i = 1,2,\dots,n; \\ 10k-5, if\ i = 5k-3, k = 1,2,\dots,n; \\ 10k-3,, if\ i = 5k-2, k = 1,2,\dots,n; \\ 10k-1, if\ i = 5k-1, k = 1,2,\dots,n; \\ 10k-7, if\ i = 5k, k = 1,2,\dots,n. \end{cases}$$

Thus, all the edge labels are distinct. Hence $PS_n$ admits vertex even mean labeling. Thus, it is vertex even mean graph.

**ILLUSTRATION 2.4:** Vertex even mean labeling of $PS_2$ is shown in figure below.

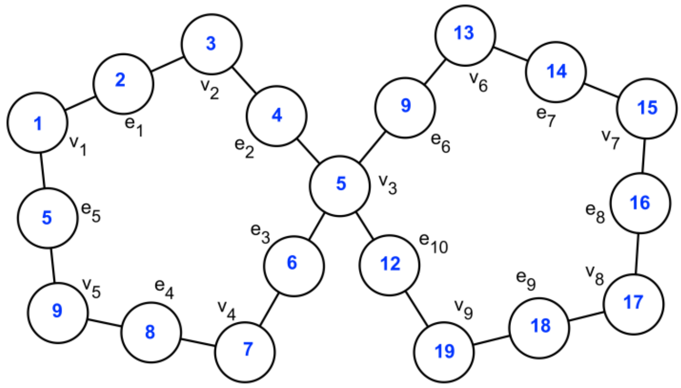


*Figure 2: Vertex odd mean labeling of $PS_2$*

**COROLLARY 2.5:** Every even vertex odd mean graph is vertex even mean graph.

**PROOF:** A graph $G$ is said to have an even vertex odd mean labeling if there exists an injective function $f: V(G) \rightarrow \{0, 2, 4, \dots, 2q\}$ such that the induced map $f^*: E(G) \rightarrow \{1, 3, 5, \dots, 2q-1\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ is a bijection. Clearly, $\{1, 3, 5, \dots, 2q-1\}$ is subset of $N$ which is the range of induced labeling $f^*$. Thus any graph which admits even vertex odd mean labeling will also admit vertex even mean labeling. Hence, the graph is vertex even mean graph.

**THEOREM 2.6:** An alternating quadrilateral snake which begins with a cycle admits vertex even and odd mean labeling.

**PROOF:** Let $G = A(QS_n)$ bean alternating quadrilateral snake graph with path $P_n$. We label the vertices of $G$ such that all the consecutive vertex lying on the path are $v_1, v_3, v_5, \dots, v_{2n-1}, v_{2n+1}$ and the vertices which does not lie on path are labeled with $v_2, v_4, v_6, \dots, v_{2n}, v_{2n+2}$. We label the edges of $G$ as $e_{4i-3}, e_{2i}, e_{4i}$ and $e_{4i-1}$ joining the vertices $v_{4i-3}v_{4i-2}, v_{4i-3}v_{4i-1}, v_{4i-2}v_{4i}$ and $v_{4i-1}v_{4i}$ respectively for $i = 1, 2, 3, \dots, \left\lceil \frac{n}{2} \right\rceil$. We label the edges which connect two cycles in $G$ with $v_{4i-1}v_{4i+1} = f_i, i = 1, 2, 3, \dots, \left\lfloor \frac{n}{2} \right\rfloor$. Thus $|E(G)| = |V(G)| = \frac{5n+3}{2}$ when $n$ is odd and $\frac{5n}{2}$ when $n$ is even.

- Case 1: Define a function f: V(G) → {1, 3, 5, … , 2q − 1} as follows:

$$f(v_i) = \begin{cases} 8k-5, & if\ i = 4k-3, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-7, & if\ i = 4k-2, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-1, & if\ i = 4k-1, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-3, & if\ i = 4k, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil. \end{cases}$$

Using above vertex labeling we obtain an induced labeling $f^*: E(G) \to N$ such that $f^*(uv) = \frac{f(u)+f(v)}{2}$ gives us:

$$f^*(e_i) = \begin{cases} 8k-6, & if\ i = 4k-3, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-3, & if\ i = 4k-2, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-2, & if\ i = 4k-1, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-5, & if\ i = 4k, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil. \end{cases}$$

$$f^*(f_i) = 8i+1, if\ i = 1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, all the edge labels are distinct. Hence $A(QS_n)$ admits vertex odd mean labeling. Thus, it is vertex odd mean graph.

- Case 2: Define a function f: V(G) → {2, 4, 6, … , 2q} as follows:

$$f(v_i) = \begin{cases} 8k-4, & if\ i = 4k-3, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-6, & if\ i = 4k-2, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k, & if\ i = 4k-1, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-2, & if\ i = 4k, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil. \end{cases}$$

Using above vertex labeling we obtain an induced labeling $f^*: E(G) \to N$ such that $f^*(uv) = \frac{f(u)+f(v)}{2}$ gives us:

$$f^*(e_i) = \begin{cases} 8k-5, & if\ i = 4k-3, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-2, & if\ i = 4k-2, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-1, & if\ i = 4k-1, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil; \\ 8k-4, & if\ i = 4k, k = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil. \end{cases}$$

$$f^*(f_i) = 8i+2, if\ i = 1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, all the edge labels are distinct. Hence $A(QS_n)$ admits vertex even mean labeling. Thus, it is vertex even mean graph.

**ILLUSTRATION 2.7:** Vertex even mean labeling of $A(QS_4)$ is shown in figure below.

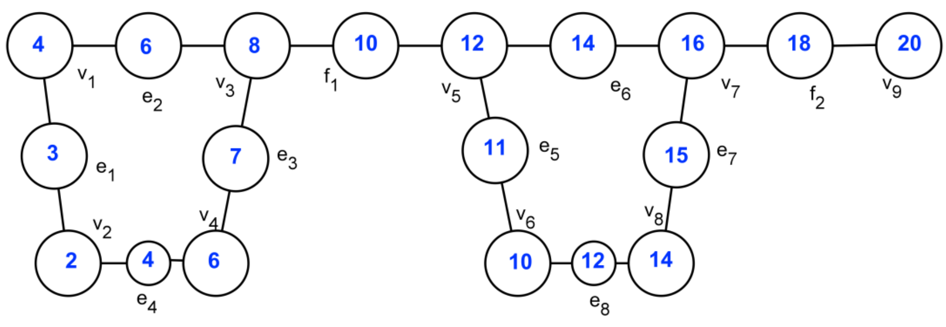


*Figure 3: Vertex even mean labeling of $A(QS_4)$*

**THEOREM 2.8:** An alternating quadrilateral snake which begins with an edge admits vertex even and odd mean labeling.

**PROOF:** Let $G = A(QS_n)$ bean alternating quadrilateral snake graph with path $P_n$. We label the vertices of $G$ such that all the consecutive vertex lying on the path are $v_1, v_3, v_5, \dots, v_{2n-1}, v_{2n+1}$ and the vertices which does not lie on path are labeled with $v_2, v_4, v_6, \dots, v_{2n}, v_{2n+2}$. We label the edges of $G$ as $e_{4i-3}, e_{2i}, e_{4i}$ and $e_{4i-1}$ joining the vertices $v_{4i-3}v_{4i-2}, v_{4i-3}v_{4i-1}, v_{4i-2}v_{4i}$

and $v_{4i-1}v_{4i}$ respectively for $i = 1,2,3,\ldots,\left\lfloor\frac{n}{2}\right\rfloor$. We label the edges which connect two cycles in $G$ with $v_{4i-3}v_{4i-1} = f_i, i = 1,2,3,\ldots,\left\lceil\frac{n}{2}\right\rceil$. Thus $|E(G)| = |V(G)| = \frac{5n-3}{2}$ when $n$ is odd and $\frac{5n}{2}$ when $n$ is even.

- Case 1: Define a function f: V(G) → {1, 3, 5, … , 2q − 1} as follows:

$$f(v_i) = \begin{cases} 1, if\ i = 1; \\ 5, if\ i = 3; \\ 8k-7, & if\ i = 4k-3, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k-5, & if\ i = 4k-2, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k-3, & if\ i = 4k-1, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k-1, & if\ i = 4k, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor. \end{cases}$$

Using above vertex labeling we obtain an induced labeling $f^*: E(G) \to N$ such that $f^*(uv) = \frac{f(u)+f(v)}{2}$ gives us:

$$f^*(f_i) = 8i-5, if\ i = 1,2,3,\ldots,\left\lceil\frac{n}{2}\right\rceil.$$

$$f^*(e_i) = \begin{cases} 8k-4, & if\ i = 4k-3, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k-1, & if\ i = 4k-2, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k, & if\ i = 4k-1, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k-3, & if\ i = 4k, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor. \end{cases}$$

Thus, all the edge labels are distinct. Hence $A(QS_n)$ admits vertex odd mean labeling. Thus, it is vertex odd mean graph.

- Case 2: Define a function f: V(G) → {2, 4, 6, … , 2q} as follows:

$$f(v_i) = \begin{cases} 2, if\ i = 1; \\ 6, if\ i = 3; \\ 8k-2, & if\ i = 4k-1, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k+2, & if\ i = 4k+1, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k-4, & if\ i = 4k-2, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k, & if\ i = 4k, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor. \end{cases}$$

Using above vertex labeling we obtain an induced labeling $f^*: E(G) \to N$ such that $f^*(uv) = \frac{f(u)+f(v)}{2}$ gives us:

$$f^*(f_i) = 8i-4, if\ i = 1,2,3,\ldots,\left\lceil\frac{n}{2}\right\rceil.$$

$$f^*(e_i) = \begin{cases} 8k-3, & if\ i = 4k-3, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k, & if\ i = 4k-2, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k+1, & if\ i = 4k-1, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor; \\ 8k-2, & if\ i = 4k, k = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor. \end{cases}$$

Thus, all the edge labels are distinct. Hence $A(QS_n)$ admits vertex even mean labeling. Thus, it is vertex even mean graph.

## III. Conclusion

We conclude that quadrilateral snake, pentagonal snake, alternating quadrilateral snake are vertex even and odd mean graph. We have also proved that any even vertex odd mean graph is vertex even mean graph. Further investigations, using different families of graphs can be carried out using various vertex and edge labelings.

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