Generalized Identities of Triple Sequence using Combinatorics

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Abstract: In this paper a we have established and proved new generalised properties on one of the schemes of multiplicative Triple sequence using combinatorics.

1. Introduction

Sequence and series have wide applications, combinatorics is a strong concept of Number theory in mathematics with the help of combinatorics many problems on mathematics have been solved. Many mathematicians have generalised many properties on well-known Fibonacci and Lucas sequence using combinatorics. The concept triple sequence was first introduced by Jin-Zai Lee & Jia-Sheng Lee [1] in 1987. There are different schemes possible for multiplicative triple sequence, in this paper we have established and prove new generalised identities by using combinatorics approach

2. Multiplicative Triple sequence

The one of the schemes of Multiplicative Triple sequence is defined by the recurrence relations

$$\alpha_{n+2} = \gamma_{n+1}\gamma_n\,, \qquad \beta_{n+2} = \alpha_{n+1}\alpha_n, \qquad \gamma_{n+2} = \beta_{n+1}\beta_n \tag{2.1}$$

For all integer $n \geq 0$, with initial conditions

$$\alpha_0 = a,\ \ \alpha_1 = d,\ \ \beta_0 = b,\ \ \beta_1 = e,\ \ \gamma_0 = c,\ \ \gamma_1 = f$$

Where $a,\ d,\ b,\ e,\ c$ and f are real numbers

Theorem 2.1 If α_n and γ_n are define by equation (2.1) then (for $n \geq 0$)

$$\gamma_{n+6m-2} = \prod_{i=n}^{n+3m-1} \alpha_i^{\binom{3m-1}{i-n}} \tag{2.2}$$

Proof: Theorem can be proved by mathematical induction method on n and m

For $n = 1$ and $m = 1$ by equations (2.1) and (2.2) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=1}^{3} \alpha_i^{\binom{2}{i-1}} = \alpha_1\alpha_2^2\alpha_3$$

by using equation (2.1) we have

$$\prod_{i=1}^{3} \alpha_i^{\binom{2}{i-1}} = \beta_3\beta_4 = \gamma_3$$

which proves for $n = 1$ and $m = 1$

Suppose the theorem is true for $n = k$ and $m = 1$ so by equation (2.2)

$$\gamma_{k+4} = \prod_{i=k}^{k+2} \alpha_i^{\binom{2}{i-k}} \tag{2.3}$$

Now to prove for $n = k + 1$ and $m = 1$ by using equation (2.1), (2.2) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=k+1}^{k+3} \alpha_i^{\binom{2}{i-(k+1)}} = \alpha_{k+1}\alpha_{k+2}^2\alpha_{k+3}$$

by using equation (2.1) we have

$$\prod_{i=k+1}^{k+3} \alpha_i^{\binom{2}{i-(k+1)}} = \gamma_{k+5}$$

which proves the theorem for $n = k + 1$ and $m = 1$.

Suppose the theorem is true for all integers $n = h$ and $m = k$ so by equation (2.2)

$$\gamma_{h+6k-2} = \prod_{i=h}^{h+3k-1} \alpha_i^{\binom{3k-1}{i-h}} \tag{2.4}$$

Now to prove for all integers $n = h$ and $m = k + 1$ by using equation (2.1), (2.2) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=h}^{h+3(k+1)-1} \alpha_i^{\binom{3(k+1)-1}{i-h}} = \alpha_h^{\binom{3k+2}{0}} \alpha_{h+1}^{\binom{3k+2}{1}} \dots \alpha_{h+3k+2}^{\binom{3k+2}{3k+2}}$$

by using equation (2.1) we have

$$\prod_{i=h}^{h+3(k+1)-1} \alpha_i^{\binom{3(k+1)-1}{i-h}} = \gamma_{h+6(k+1)-2}$$

which proves the theorem.

Theorem 2.2 If β_n and γ_n are define by equation (2.1) then (for $n \geq 0$)

$$\beta_{n+6k-2} = \prod_{i=n}^{n+3k-1} \gamma_i^{\binom{3k-1}{i-n}} \tag{2.5}$$

Proof: Theorem can be proved by mathematical induction method on n and m

For $n = 1$ and $m = 1$ by equations (2.1) and (2.5) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=1}^{3} \gamma_i^{\binom{2}{i-1}} = \gamma_1\gamma_2^2\gamma_3$$

by using equation (2.1) we have

$$\prod_{i=1}^{3} \gamma_i^{\binom{2}{i-1}} = \alpha_3\alpha_4 = \beta_3$$

which proves for $n = 1$ and $m = 1$

Suppose the theorem is true for $n = k$ and $m = 1$ so by equation (2.5)

$$\beta_{k+4} = \prod_{i=k}^{k+2} \gamma_i^{\binom{2}{i-k}} \tag{2.6}$$

Now to prove for $n = k + 1$ and $m = 1$ by using equation (2.1), (2.5) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=k+1}^{k+3} \gamma_i^{\binom{2}{i-(k+1)}} = \gamma_{k+1}\gamma_{k+2}^2\gamma_{k+3}$$

by using equation (2.1) we have

$$\prod_{i=k+1}^{k+3} \gamma_i^{\binom{2}{i-(k+1)}} = \beta_{k+5}$$

which proves the theorem for $n = k + 1$ and $m = 1$.

Suppose the theorem is true for all integers $n = h$ and $m = k$ so by equation (2.5)

$$\beta_{h+6k-2} = \prod_{i=h}^{h+3k-1} \gamma_i^{\binom{3k-1}{i-h}} \tag{2.7}$$

Now to prove for all integers $n = h$ and $m = k + 1$ by using equation (2.1), (2.5) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=h}^{h+3(k+1)-1} \gamma_i^{\binom{3(k+1)-1}{i-h}} = \gamma_h^{\binom{3k+2}{0}}\gamma_{h+1}^{\binom{3k+2}{1}} \dots \gamma_{h+3k+2}^{\binom{3k+2}{3k+2}}$$

by using equation (2.1) we have

$$\prod_{i=h}^{h+3(k+1)-1} \gamma_i^{\binom{3(k+1)-1}{i-h}} = \beta_{h+6(k+1)-2}$$

which proves the theorem.

Theorem 2.3 If α_n and β_n are define by equation (2.1) then (for $n \geq 0$)

$$\alpha_{n+6k-2} = \prod_{i=n}^{n+3k-1} \beta_i^{\binom{3k-1}{i-n}} \tag{2.8}$$

Proof: Theorem can be proved by mathematical induction method on n and m

For $n = 1$ and $m = 1$ by equations (2.1) and (2.8) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=1}^{3} \beta_i^{\binom{2}{i-1}} = \beta_1\beta_2^2\beta_3$$

by using equation (2.1) we have

$$\prod_{i=1}^{3} \beta_i^{\binom{2}{i-1}} = \gamma_3\gamma_4 = \alpha_3$$

which proves for $n = 1$ and $m = 1$

Suppose the theorem is true for $n = k$ and $m = 1$ so by equation (2.8)

$$\alpha_{k+4} = \prod_{i=k}^{k+2} \beta_i^{\binom{2}{i-k}} \tag{2.9}$$

Now to prove for $n = k + 1$ and $m = 1$ by using equation (2.1), (2.8) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=k+1}^{k+3} \beta_i^{\binom{2}{i-(k+1)}} = \beta_{k+1}\beta_{k+2}^2\beta_{k+3}$$

by using equation (2.1) we have

$$\prod_{i=k+1}^{k+3} \beta_i^{\binom{2}{i-(k+1)}} = \alpha_{k+5}$$

which proves the theorem for $n = k + 1$ and $m = 1$.

Suppose the theorem is true for all integers $n = h$ and $m = k$ so by equation (2.8)

$$\alpha_{h+6k-2} = \prod_{i=h}^{h+3k-1} \beta_i^{\binom{3k-1}{i-h}} \tag{2.10}$$

Now to prove for all integers $n = h$ and $m = k + 1$ by using equation (2.1), (2.8) and the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\prod_{i=h}^{h+3(k+1)-1} \beta_i^{\binom{3(k+1)-1}{i-h}} = \beta_h^{\binom{3k+2}{0}}\beta_{h+1}^{\binom{3k+2}{1}} \dots \beta_{h+3k+2}^{\binom{3k+2}{3k+2}}$$

by using equation (2.1) we have

$$\prod_{i=h}^{h+3(k+1)-1} \beta_i^{\binom{3(k+1)-1}{i-h}} = \alpha_{h+6(k+1)-2}$$

which proves the theorem.

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