Solution of 1-D Coupled Keller-Segel equations Using Homotopy Perturbation Transformation Method

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Abstract

The nonlinear Keller-Segel equations have been solved by homotopy perturbation transformation method (HPTM) which is the coupling of Laplace transform method and the homotopy perturbation method. He's polynomials can be handily applied over the nonlinear term present in the differential equation. The result shows that the HPTM is a very efficient and simple method that is free from round-off errors. Two test examples are considered to illustrate the present scheme. Further, the results are compared with different methods reported in the literature.

Keywords: Homotopy perturbation method; Laplace transform method; He's polynomials; Keller-Segel equation.

1. Introduction

Nonlinear phenomena's are of keen interest of mathematicians, physicists, engineersand many scientists as mostly systems in the nature are inherently nonlinear.Mostly nonlinear systems are difficult to solve as we have to approximate such problems by use of linear equations and often it is more difficult to get an analytic solution for such problems. Various methods were proposed to find approximate solutionsof nonlinear equations. A combination of the homotopy in topology and some classical perturbation techniques (homotopy perturbation method (HPM)) was developed by He, which has been further applied to solve a lots of linear and nonlinear differential equations [1-4]. In the recent years, a highly effective technique was developed by combining homotopy perturbation method with Laplace transformation method and the variational iteration method to handle the nonlinear terms is known as homotopy perturbation transform method (HPTM). This method provides the solution in rapid convergent series which leads the solution in a closed form. The use of He's polynomials in the nonlinear terms was first introduced by Ghorbani [5]. Later on many researcher use homotopy perturbation transform method for different type of linear and nonlinear differential equations [6-7]. In this paper, HPTM is applied to find the solution ofcoupled attractor for one dimensional Keller Siegel equation. This equation has been formed to various numbers of physical phenomena [8]. Many researchers used different techniques to solve one dimensional Keller Siegel equation [9-10]. Present study shows that HPTM is highly efficient for solving nonlinear equations.

2. Homotopy perturbation transform method

To elucidate the basic idea of this method, we consider coupled attractor for one-dimensional Keller-Segel equation:

$$U_t(x,t) = aU_{xx}(x,t) - (U(x,t)\chi_x(\rho))_x$$

$$\rho_t(x,t) = b\rho_{xx} + cU(x,t) - d\rho(x,t) \quad (1)$$

Subjected to initial condition:

$$U(x,0) = U_0(x), \rho(x,0) = \rho_0(x) \quad (2)$$

Taking Laplace transform on both sides of equation (1) and applying the differentiation property, we have

$$U(x,s) = \frac{U(x,0)}{s} + \frac{1}{s}\,\mathcal{L}\,[aU_{xx}(x,t) - (U(x,t)\chi_x(\rho))_x] \quad (5)$$

$$\rho(x,s) = \frac{\rho(x,0)}{s} + \frac{1}{s}\,\mathcal{L}\,[b\rho_{xx} + U(x,t) - d\rho(x,t)] \quad (6)$$

Taking the inverse Laplace transform on both sides of equation (5) and (6)

$$U(x,t) = U(x,0) + \mathcal{L}^{-1}\left\{\frac{1}{s}\,\mathcal{L}\,[aU_{xx}(x,t) - (U(x,t)\chi_x(\rho))_x]\right\} \quad (7)$$

$$\rho(x,t) = \rho(x,0) + \mathcal{L}^{-1}\left\{\frac{1}{s}\,\mathcal{L}\,[b\rho_{xx} + U(x,t) - d\rho(x,t)]\right\} \quad (8)$$

and now apply homotopy perturbation method, with

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), NU(x,t) = \sum_{n=0}^{\infty} p^n H_n(U)$$

$$\rho(x,t) = \sum_{n=0}^{\infty} p^n \rho_n(x,t) \quad (9)$$

Where $H_n(U)$ is He's polynomial use to decompose the nonlinear terms [8].

Substituting equation (9) in equation (7) and (8), we get

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = U(x,0) + p\mathcal{L}^{-1}\left\{\frac{1}{s}\,\mathcal{L}\,[a\sum_{n=0}^{\infty}(U_n p^n)_{xx} - \sum_{n=0}^{\infty}(H_n p^n)]\right\} \quad (10)$$

$$\sum_{n=0}^{\infty} p^n \rho_n(x,t) = \rho(x,0) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b(\sum_{n=1}^{\infty} p^n \rho_n(x,t))_{xx} + c(\sum_{n=1}^{\infty} p^n U_n(x,t)) - d\sum_{n=0}^{\infty} p^n \rho_n(x,t)\}\right] \quad (11)$$

Comparing the coefficients of like powers of p, the following approximations are obtained

$$p^0:\ U_0 = U(x,0),\ \ \rho_0 = \rho(x,0)$$

$$p^1: U_1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[a\,U_{0xx} - H_0]\right\}, \rho_1 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b\rho_{0xx} + cU_0 - d\rho_0\}\right],$$

$$p^2: U_2 = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[a\,U_{1xx} - H_1]\right\}, \rho_2 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b\rho_{1xx} + cU_1 - d\rho_1\}\right],$$

$$\vdots$$

(12)

Setting, $p = 1$ results the approximate solution of equation (1)

$$U(x,t) = U_0 + U_1 + U_2 + \cdots, \quad \rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \cdots, \tag{13}$$

3. Applications

In order to understand the solution procedure of the homotopy perturbation transform method,we consider the following two examples:

Example 3.1: Consider the Keller Siegel equation (1) with sensitivity function $\chi(\rho) = \rho^2$. Then the Chemo-tactic term i.e. $\frac{\partial}{\partial x}\left(U(x,t)\frac{\partial\chi(\rho)}{\partial x}\right) = 2\rho\frac{\partial\rho}{\partial x}\frac{\partial U}{\partial x} + 2U\rho\frac{\partial^2\rho}{\partial x^2} + 2U\left(\frac{\partial\rho}{\partial x}\right)^2$

Eq. (1) become $\frac{\partial U(x,t)}{\partial t} = a\,\frac{\partial^2 U(x,t)}{\partial x^2} - \left(2\rho\frac{\partial\rho}{\partial x}\frac{\partial U}{\partial x} + 2U\rho\frac{\partial^2\rho}{\partial x^2} + 2U\left(\frac{\partial\rho}{\partial x}\right)^2\right)$,

and $\frac{\partial\rho(x,t)}{\partial t} = b\frac{\partial^2\rho(x,t)}{\partial x^2} + cU(x,t) - d\rho(x,t)$, subject to the initial conditions

$U(x,0) = m\sin x$ and $\rho(x,0) = n\sin x,\ x > 0$

subjected to the initial conditions

$$U(x,0) = m\sin x, \rho(x,0) = n\sin x, x > 0 \tag{16}$$

By HPTM, we have

$$\sum_{n=1}^{\infty} p^n U_n(x,t) = U(x,0) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{a(\textstyle\sum_{n=1}^{\infty} p^n U_n(x,t))_{xx} - \sum_{n=1}^{\infty} p^n H_n\}\right] \tag{17}$$

$$\sum_{n=1}^{\infty} p^n \rho_n(x,t) = \rho(x,0) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b(\textstyle\sum_{n=1}^{\infty} p^n \rho_n(x,t))_{xx} + c(\sum_{n=1}^{\infty} p^n U_n(x,t)) - d\sum_{n=1}^{\infty} p^n \rho_n(x,t)\}\right] \tag{18}$$

where $\sum_{n=1}^{\infty} p^n H_n = NU(x,t)$

$$= 2\left(\sum_{n=1}^{\infty} p^n\rho_n(x,t)\right)\left(\sum_{n=1}^{\infty} p^n\rho_n(x,t)\right)_x\left(\sum_{n=1}^{\infty} p^n U_n(x,t)\right)_x + 2\left(\sum_{n=1}^{\infty} p^n U_n(x,t)\right)\left(\sum_{n=1}^{\infty} p^n\rho_n(x,t)\right)\left(\sum_{n=1}^{\infty} p^n\rho_n(x,t)\right)_{xx} + 2\left(\sum_{n=1}^{\infty} p^n U_n(x,t)\right)\left(\left(\sum_{n=1}^{\infty} p^n\rho_n(x,t)\right)_x\right)^2$$

On comparing the like power of p on both sides and by imposing the initial conditions, we get

On comparing the like power of p on both sides ,we get

$U_0 = U(x,0)$, $\rho_0 = \rho(x,0)$,

$U_1 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{aU_{0xx} - H_0\}\right]$, $\rho_1 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b\rho_{0xx} + cU_o - d\rho_o\}\right]$,

,$U_2 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{aU_{1xx} - H_1\}\right]$, $\rho_2 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b\rho_{1xx} + cU_1 - d\rho_1\}\right]$,

$U_3 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{aU_{2xx} - H_2\}\right], \rho_3 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b\rho_{2xx} + cU_2 - d\rho_2\}\right]$

$$\vdots$$

where

$$H_o = 2\rho_0 U_{0x}\,\rho_{0x} + 2U_0\rho_0\rho_{0xx} + 2U_0\rho_{0x}^2$$

$$H_1 = 2[\rho_0(U_{0x}\,\rho_{1x} + U_{1x}\,\rho_{0x}) + \rho_1(U_{0x}\,\rho_{0x})] + 2(U_0\rho_0\rho_{1xx} + (U_0\rho_1 + U_1\rho_0)\rho_{0xx}) + 2(U_1\rho_{0x}^2 + 2U_{0x}\rho_{0x}\rho_{1x})$$

$$U_0 = m\sin x\,,$$

$$U_1 = -mt\sin x\,(3n^2\cos 2x + (a+n^2))$$

$$U_2 = m\frac{t^2}{2}\sin x[27an^2 + (a^2 + 13\,an^2) - (2\cos 2x + 1)\{2(cm - (b+d)n - an)n\} + n^4(6\cos 4x - 3\cos 2x + 17)]$$

$$\vdots \tag{22}$$

$$\rho_0 = n\sin x\,,$$

$$\rho_1 = t\sin x\,(cm - (b+d)n)$$

$$\rho_2 = -\frac{t^2}{2}\sin x\,[(cm - (b+d)n)(b+d) + cm(3n^2\cos 2x + (a+n^2))]$$

$$\vdots \tag{23}$$

Hence the solution of eq.(1) is obtained as $p \to 1$ .i.e.

$$U(x,t) = U_0 + U_1 + U_2 + \cdots, \quad \rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \cdots, \tag{24}$$

$$U(x,t) = m\sin x - mt\sin x\,(3n^2\cos 2x + (a+n^2)) + m\frac{t^2}{2}\sin x[27an^2 + (a^2 + 13\,an^2) - (2\cos 2x + 1)\{2(cm - (b+d)n - an)n\} + n^4(6\cos 4x - 3\cos 2x + 17)] + \cdots$$

$$\rho(x,t) = n\sin x + t\sin x\,(cm - (b+d)n) - \frac{t^2}{2}\sin x\,[(cm - (b+d)n)(b+d) + cm(3n^2\cos 2x + (a+n^2))] + \cdots$$

Example 3.2: Consider the Keller Siegel equation with sensitivity function $\chi(\rho) = \rho$. Then the Chemo-tactic term i.e. $\frac{\partial}{\partial x}\left(U(x,t)\frac{\partial\chi(\rho)}{\partial x}\right) = \frac{\partial}{\partial x}\left(U(x,t)\frac{\partial\rho(x,t)}{\partial x}\right) = \frac{\partial U}{\partial x}\frac{\partial\rho}{\partial x} + U\frac{\partial^2\rho}{\partial x^2}$

Fractional attractor one dimensional Keller Siegel equation

$$\frac{\partial U(x,t)}{\partial t} = a\,\frac{\partial^2 U(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(U(x,t)\frac{\partial \chi(\rho)}{\partial x}\right)$$

$$\text{and } \frac{\partial \rho(x,t)}{\partial t} = b\,\frac{\partial^2 \rho(x,t)}{\partial x^2} + cU(x,t) - d\rho(x,t) \tag{25}$$

Eq. (25) become $\frac{\partial U(x,t)}{\partial t} = a\,\frac{\partial^2 U(x,t)}{\partial x^2} - \left(\frac{\partial U}{\partial x}\frac{\partial \rho}{\partial x} + U\frac{\partial^2 \rho}{\partial x^2}\right)$,

and $\frac{\partial \rho(x,t)}{\partial t} = b\,\frac{\partial^2 \rho(x,t)}{\partial x^2} + cU(x,t) - d\rho(x,t)$, subject to the initial conditions

$U(x,0) = me^{-x}$ and $\rho(x,0) = ne^{-x}$,

$$U(x,0) = me^{-x},\ \rho(x,0) = ne^{-x}, \tag{26}$$

By HPTM, we have

$$\sum_{n=1}^{\infty} p^n U_n(x,t) = U(x,0) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{a(\sum_{n=1}^{\infty} p^n U_n(x,t))_{xx} - \sum_{n=1}^{\infty} p^n H_n\}\right]$$

$$\sum_{n=1}^{\infty} p^n \rho_n(x,t) = \rho(x,0) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b(\sum_{n=1}^{\infty} p^n \rho_n(x,t))_{xx} + c(\sum_{n=1}^{\infty} p^n U_n(x,t)) - d\sum_{n=1}^{\infty} p^n \rho_n(x,t)\}\right] \tag{27}$$

On comparing the like power of p on both sides ,we get

$U_0 = U(x,0)$, $U_1 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{aU_{0xx} - H_0\}\right]$, $U_2 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{aU_{1xx} - H_1\}\right]$

$\rho_0 = \rho(x,0)$, $\rho_1 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b\rho_{0xx} + cU_o - d\rho_o\}\right]$, $\rho_2 = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{b\rho_{1xx} + cU_1 - d\rho_1\}\right]$

$$\vdots$$

where $\sum_{n=1}^{\infty} p^n H_n = NU(x,t)$

$= (\sum_{n=1}^{\infty} p^n \rho_n(x,t))_x(\sum_{n=1}^{\infty} p^n U_n(x,t))_x + (\sum_{n=1}^{\infty} p^n \rho_n(x,t))_{xx}(\sum_{n=1}^{\infty} p^n U_n(x,t))$

$H_o = U_{0x}\,\rho_{0x} + U_0\rho_{0xx}$

$H_1 = (U_{0x}\,\rho_{1x} + U_{1x}\,\rho_{0x}) + (U_0\rho_{1xx} + U_1\rho_{0xx})$

$H_2 = (U_{0x}\,\rho_{2x} + U_{1x}\,\rho_{1x} + U_{2x}\,\rho_{0x}) + (U_0\rho_{2xx} + U_1\rho_{1xx} + U_2\rho_{0xx})$

$$\vdots$$

$$U_0 = me^{-x},$$

$$U_1 = mte^{-2x}(ae^x - 2n),$$

$$U_2 = me^{-2x}\frac{t^2}{2}[a^2e^x - 2(cm + (b-d)n) - 10an + 6n^2e^{-x}],$$

$$U_3 = me^{-2x}\frac{t^3}{6}[a^3e^x + 24\,an^2e^{-x} - 40a^2n + 2(cm + (b-d)n)(b - d - 2a - 9ne^{-x}) + 2cm(a - ne^{-x})]$$

$$\vdots \tag{28}$$

$$\rho_0 = ne^{-x},$$
$$\rho_1 = e^{-x}t\,(cm + (b-d)n),$$
$$\rho_2 = e^{-x}\frac{t^2}{2}[(cm + (b-d)n)(b-d) + cm(a - 2ne^{-x})],$$
$$\rho_3 = e^{-x}\frac{t^3}{6}\begin{bmatrix}(cm + (b-d)n)(b-d)^2 + acm(b-d) - 2cmne^{-x}(4b-d) + c^2a^2m \\ +e^{-2x}cm(54n^2 - 8(cm + (b-d)n))\end{bmatrix}$$
$$\vdots$$

(29)

Hence the solution of eq.(25) using eq. (28) and (29) is obtained as $p \to 1$ .i.e.

$U(x,t) = U_0 + U_1 + U_2 + \cdots,\quad \rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \cdots,$

4. Conclusion

In this work, homotopy perturbation transform method (HPTM) has been successfully applied to approximate solution for a system of nonlinear partial differential equations derived from an attractor for a one-dimensional Keller-Segel dynamics system. On comparing the results of thismethod with HPM, it is observed HPTM is extremely simple, straight forward and easy to handle the nonlinear terms.The main advantage of this method is to overcome the lack of satisfied initial conditions and to construct homotopy, which is a difficult task in case of HPM. Further, the method needs much less computational work which shows fast convergent for solving nonlinear system of partial differential equations.

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