Investigation of a system having *k*-out-of-*n* and standby sub systems

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Abstract

Redundancy is the technique which is generally used to enhance the system performance at both the levels (according to the requirement of the system) i.e. at system level as well as unit level. The aim of this paper is to investigate a system which is having a *k*-out-of-*n*: F (for specific value of *k* and *n*) unit along with a standby unit with perfect switching i.e. when the *k*-out-of-*n*: F unit of the system is failed then standby unit take charge with the help of a switch which automatically turns the standby unit when get a signal that main unit of the system is failed. The study is carried out by developing a mathematical model for the considered system with the help of Makov birth-death process and then solve the developed model by using supplementary variable technique and Laplace transformation to draw some important system's performance characteristics namely availability, reliability, MTTF and sensitivity analysis. Graphs are drawn to get more insights and clearly about the obtained results.

Key words: *k*-out-of-*n*: F; standby unit; Makov birth-death process; supplementary variable technique; system's performance characteristics;

1. Introduction

It can be seen for the past research work [1-4] that reliability analysis for different industrial systems is foremost useful to enhance the performance of the same. More over many industries use different redundancies technique to improve its productivity and reliability. Elite works were done by focusing on *k*-out-of-*n* redundancy. Gemund and Reijns [5] presented an analytical methodology to evaluate the mean residual time of a *k*-out-of-*n* system with a single cold standby component with the aid of Pearson distribution. Coit [6] developed a methodology to maximize system reliability for a non-repairable system by the aid of cold standby redundancy. The developed methodology is helpful to identify imperfect component failure along with imperfect switching degradation. Authors including [7-10] investigates standby redundancy to increase the reliability/availability of various system by taking the different aspects e.g. perfect/ imperfect switching. Based on the above literature, in the present paper author investigated a system which is having a k-out-of-n system along with a standby unit. The study is carried out by developing a mathematical model for the considered system with the help of Makov birth-death process and then solve the developed model by using supplementary variable technique and Laplace transformation to draw some important system's performance characteristics. The following Fig.1 gives the various possible state of the considered system (according to different failure and repairs) in which it takes place in a time interval Δt.

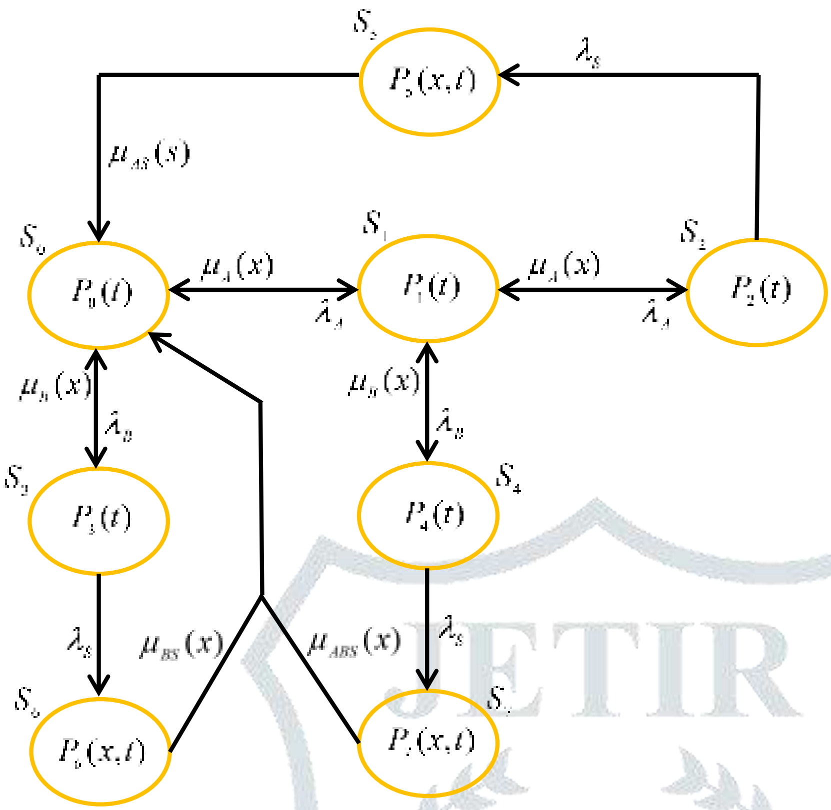

Fig.1 State transition diagram

2. Assumptions

- The switching device which switches the system from main unit to standby unit is perfect.
- The transition time from main unit to standby unit is negligible.
- Simultaneous failure is not considered.
- Perfect maintenance is available every time.
- The developed system is solved for constant failure and repair rates.

3. Notations and states description

The brief description about the various notations and states used in paper is given in the following Table 1 and Table 2.

$P_i(t)$; $i=0,1,2,3,4$	The state probability of the system being in state $S_i; i=0,1,2,3,4$.
$\overline{P}_i(s)$	Laplace transform of the state $P_i(t)$.
$P_i(x,t)$; $i=5,6,7$	The state probability of the system being in failed state $S_i; i=5,6,7$.
$\lambda_A/\lambda_B/\lambda_S$	The failure rates of the unit A /B/standby unit.

$\mu_A(x)/\mu_B(x)/\mu_S(x)$	Repair rate of A /B/standby unit.
$\mu_{AS}(x)/\mu_{BS}(x)/\mu_{ABS}(x)$	Simultaneous repair rate of unit A with standby unit/ unit B with standby unit /Unit A, Unit B with standby.
t/s	Time unit/Laplace transform variable.

Table 1 Notations

S_0	The state in which both the Unit of the system namely A and B of the system is working.
S_1	The state in which one of the sub unit of system A is failed.
S_2	The state in which unit A is failed and system is working with standby unit.
S_3	The state in which unit B is failed and system is working with standby unit.
S_4	The state in which one of the sub unit of system A along with Unit B is failed and system is working with standby unit.
S_5	The state in which unit A along with standby unit is failed.
S_6	The state in which unit B along with standby unit is failed.
S_7	The state in which one of the sub unit system A, unit B along with standby unit is failed.

Table 2 State descriptions

4. Mathematical modelling and solution

On the basis of Markov process and Fig. 1 the following set of mathematical equation are generated.

$$\left(\frac{\partial}{\partial t}+\lambda_A+\lambda_B\right)P_0(t)=\mu_A(x)P_1(t)+\mu_B(x)P_3(t)+\sum_{i,j}\int_0^\infty \mu_i(x)P_j(x,t)\,dx$$

where $i = AS, BS, ABS;$ (1)

$j = 5,6,7$ respectively

$$\left(\frac{\partial}{\partial t}+\lambda_A+\lambda_B+\mu_A(x)\right)P_1(t)=\lambda_A\,P_0(t)+\mu_B(x)P_4(t)+\mu_A(x)P_2(t) \quad (2)$$

$$\left(\frac{\partial}{\partial t}+\lambda_S+\mu_A(x)\right)P_2(t)=\lambda_A P_1(t) \qquad (3)$$

$$\left(\frac{\partial}{\partial t}+\lambda_S+\mu_B(x)\right)P_3(t)=\lambda_B P_0(t) \qquad (4)$$

$$\left(\frac{\partial}{\partial t}+\lambda_S+\mu_B(x)\right)P_4(t)=\lambda_B P_1(t) \qquad (5)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu_{AS}(x)\right)P_5(x,t)=0 \qquad (6)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu_{BS}(x)\right)P_6(x,t)=0 \qquad (7)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu_{ABS}(x)\right)P_7(x,t)=0 \qquad (8)$$

Boundary conditions

$P_i(0,t)=\lambda_j P_k(t)$

Where $i=5,6,7;$

$j=S,S,S;$ (9)

$k=2,3,4;$ respectively

Initial conditions

$$P_0(0)=0; P_i(0)=0 \text{ for } i=1,2,3,4 \qquad (10)$$

The solution of the equations (1)-(10) can be done by using Laplace transformation. Therefore, by using Laplace transformation the above set of equation can be rewritten as follows.

$$(s+\lambda_A+\lambda_B)\overline{P}_0(s)=1+\mu_A(x)\overline{P}_1(s)+\mu_B(x)\overline{P}_3(s)+\sum_{i,j}\int_0^\infty \mu_i(x)\overline{P}_j(x,s)\,dx$$

where $i=AS,BS,ABS;$ (11)

$j=5,6,7$ respectively

$$(s+\lambda_A+\lambda_B+\mu_A(x))\overline{P}_1(s)=\lambda_A\overline{P}_0(s)+\mu_B(x)\overline{P}_4(s)+\mu_A(x)\overline{P}_2(s) \qquad (12)$$

$$(s+\lambda_S+\mu_A(x))\overline{P}_2(s)=\lambda_A\overline{P}_1(s) \qquad (13)$$

$$(s+\lambda_S+\mu_B(x))\overline{P}_3(s)=\lambda_B\overline{P}_0(s) \qquad (14)$$

$$(s+\lambda_S+\mu_B(x))\overline{P}_4(s)=\lambda_B\overline{P}_1(s) \qquad (15)$$

$$\left(\frac{\partial}{\partial x}+s+\mu_{AS}(x)\right)\overline{P}_5(x,s)=0 \qquad (16)$$

$$\left(\frac{\partial}{\partial x}+s+\mu_{BS}(x)\right)\overline{P}_6(x,s)=0 \quad (17)$$

$$\left(\frac{\partial}{\partial x}+s+\mu_{ABS}(x)\right)\overline{P}_7(x,s)=0 \quad (18)$$

Boundary conditions

$$\overline{P}_i(0,s)=\lambda_j\overline{P}_k(s)$$

Where $i=5,6,7;$

$j=S,S,S;$

$k=2,3,4;$ respectively (19)

Initial conditions

$$\overline{P}_0(0)=0;\overline{P}_i(0)=0 \text{ for } i=1,2,3,4 \quad (20)$$

The various transition state probabilities of the considered system are obtained as (solving equation (11)-(20))

$$\overline{P}_0(s)=\frac{1}{\left\{\begin{array}{l} s+\lambda_A+\lambda_B-\dfrac{1}{\left(s+\lambda_A+\lambda_B+\mu_A(x)-\dfrac{\lambda_B\mu_B(x)}{(s+\mu_B(s)+\lambda_S)}-\dfrac{\lambda_A\mu_A(x)}{(s+\mu_A(s)+\lambda_S)}\right)} \\ \left(\lambda_A\mu_A(x)+\dfrac{\lambda_A{}^2\left(1+\dfrac{\lambda_S\mu_{AS}(x)}{s+\mu_{AS}(x)}\right)}{(s+\mu_A(s)+\lambda_S)}+\dfrac{\lambda_A\lambda_B\lambda_C\mu_{ABS}(x)}{(s+\mu_{ABS}(x))(s+\mu_B(x)+\lambda_S)}\right) \\ -\dfrac{\lambda_B\lambda_S\mu_{BS}(x)}{(s+\mu_B(x)+\lambda_S)(s+\mu_{BS}(x))} \end{array}\right\}} \quad (21)$$

$$\overline{P}_1(s)=\frac{\lambda_A}{\left(s+\lambda_A+\lambda_B+\mu_A(x)-\dfrac{\lambda_B\mu_B(x)}{(s+\mu_B(s)+\lambda_S)}-\dfrac{\lambda_A\mu_A(x)}{(s+\mu_A(s)+\lambda_S)}\right)}\overline{P}_0(s) \quad (22)$$

$$\overline{P}_2(s)=\frac{\lambda_A{}^2}{(s+\mu_A(x)+\lambda_S)\left(s+\lambda_A+\lambda_B+\mu_A(x)-\dfrac{\lambda_B\mu_B(x)}{(s+\mu_B(s)+\lambda_S)}-\dfrac{\lambda_A\mu_A(x)}{(s+\mu_A(s)+\lambda_S)}\right)}\overline{P}_0(s) \quad (23)$$

$$\overline{P}_3(s)=\frac{\lambda_B}{(s+\mu_B(x)+\lambda_S)}\overline{P}_0(s) \quad (24)$$

$$\overline{P}_4(s)=\frac{\lambda_A\lambda_B}{(s+\mu_B(s)+\lambda_S)\left(s+\lambda_A+\lambda_B+\mu_A(x)-\dfrac{\lambda_B\mu_B(x)}{(s+\mu_B(s)+\lambda_S)}-\dfrac{\lambda_A\mu_A(x)}{(s+\mu_A(s)+\lambda_S)}\right)}\overline{P}_0(s) \quad (25)$$

The working state probability or availability $A(t)$ of the considered system will be given as the sum of the probabilities of good as well as degraded states (From Fig. 1) i.e.

$$\overline{A}(s) = \overline{P}_0(s) + \overline{P}_1(s) + \overline{P}_2(s) + \overline{P}_3(s) + \overline{P}_4(s) = \sum_{i=0}^{4} \overline{P}_i(s) \quad (26)$$

The various reliability measures as well as time dependent availability of the considered system can be obtained by taking inverse Laplace transformation of the equation (26).

5. Reliability measures for the considered system

5.1 Availability

Availability of a system gives the probability that the system is available for the functioning/production. For the value of time dependent availability of the considered system author takes inverse Laplace transform of equation (26) and then using the values of different failure and repair rates as $\lambda_A = 0.10,\ \lambda_B = 0.15,\ \lambda_S = 0.20,$ $\mu_A(x) = \mu_B(x) = \mu_{AS}(x) = \mu_{BS}(x) = \mu_{ABS}(x) = 1$ in obtained equation. The time dependent availability of the considered system is obtained as given in equation (27).

$$A(t) = \begin{pmatrix} 1.022399819\exp(-0.09677630169\,t) - 0.05897752772 \\ \exp(-1.117899389t)\cos(0.1564917539t) + 0.3757752 \\ \exp(-1.117899389t)\sin(0.1564917539t) + 0.04061163 \\ \exp(-0.8096975676t) - 0.004033926054\exp(-1.72772t) \end{pmatrix} \quad (27)$$

As a result, the behavior of availability w.r.t. to time unit *t* can be obtained by using (27). The same is given in Table 3 and corresponding Fig. 2.

Time Unit "*t*"	**Availability** $A(t)$
0	1.0000000000
1	0.9455700579
2	0.8567768297
3	0.7724349273
4	0.6977884004
5	0.6317349048
6	0.5727067153
7	0.5195588658
8	0.4715021569
9	0.4279567673
10	0.3884600045

Table 3 Availability vs. Time unit "*t*"

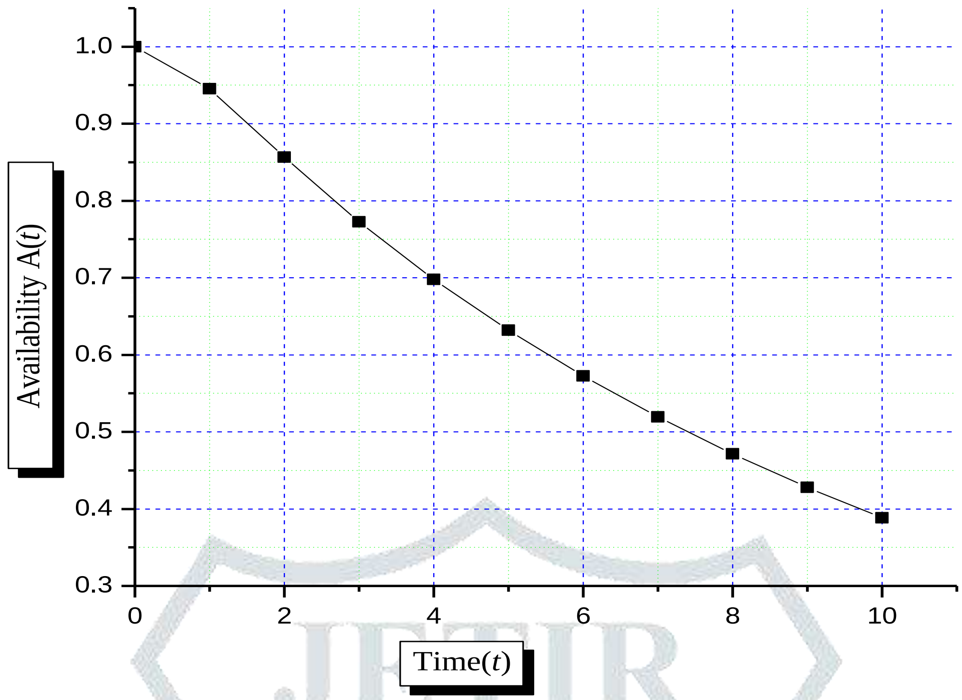

Fig. 2 Availability vs. Time "*t*"

5.2 Reliability

This is one of another measure of system performance. Here author calculate the time dependent reliability of the considered system by using equation (26) as given in equation (28).

$$R(t) = \frac{\left(\lambda_A^{\ 2} + 2\lambda_A\lambda_B + \lambda_B^{\ 2} - \lambda_B\lambda_S\right)e^{-\lambda_S t} + \left(-\lambda_A\lambda_S\left\{\lambda_A + \lambda_B - \lambda_S\right\}t - \lambda_s\left\{2\lambda_A + \lambda_B - \lambda_S\right\}\right)\, e^{-(\lambda_A+\lambda_B)t}}{\left(\lambda_A + \lambda_B - \lambda_S\right)^2} \tag{28}$$

Using different failure rates as $\lambda_A = 0.10,\ \lambda_B = 0.15,\ \lambda_S = 0.20,$ in equation (28). The time dependent reliability of the considered system is obtained as given in equation (29).

$$R(t) = 13\exp\left(-0.20\, t\right) + 400(-0.001000\, t - 0.0300)\exp\left(-0.25\, t\right) \tag{29}$$

As a result, the behavior of reliability w.r.t. to time unit *t* can be obtained by using (29). The same is given in Table 4 and corresponding Fig. 3.

Time Unit "*t*"	Reliability $R(t)$
0	1.000000000
1	0.986370078
2	0.950568154
3	0.899312773
4	0.838116133
5	0.771365580
6	0.702450449
7	0.633906167
8	0.567558429
9	0.504657643
10	0.445998704

Table 3 Reliability vs. Time unit "*t*"

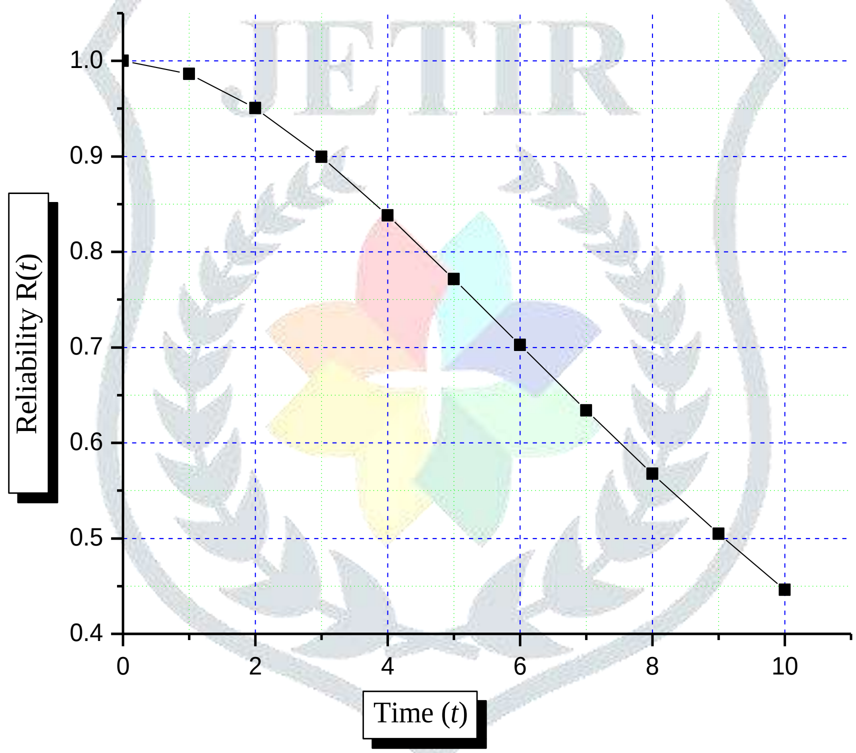

Fig. 3 Reliability vs. Time unit "*t*"

5.3 Mean Time to Failure

An essential objective for reliability designers is to measure system's lifetime against product failures. MTTF resolve this query of the system engineers/designers. MTTF is a basic lifetime metric of reliability to specify the lifetime of a system. It is calculated as

$$MTTF = \int_0^\infty R(t)\,dt \tag{30}$$

Using equation (30) the MTTF of the considered system is obtained as

$$MTTF = \frac{1}{\lambda_A + \lambda_B}\left\{1 + \frac{\lambda_A}{(\lambda_A + \lambda_B)} + \frac{\lambda_A^{\,2}}{(\lambda_A + \lambda_B)\lambda_S} + \frac{\lambda_B}{\lambda_S} + \frac{\lambda_A \lambda_B}{(\lambda_A + \lambda_B)\lambda_s}\right\} \tag{31}$$

Table 5 and corresponding Fig. 4 give the behavior of system's MTTF w.r.t. to variation in failure rates.

Failure rate	**MTTF**		
	λ_A	λ_B	λ_S
0.10	10.60000000	12.50000000	15.60000000
0.12	10.34979424	11.61157025	13.93333333
0.14	10.11296076	10.90277778	12.74285714
0.16	9.890738814	10.32544379	11.85000000
0.18	9.683195592	9.846938775	11.15555556
0.20	9.489795918	9.444444445	10.60000000

Table 5 System's MTTF vs. Failure rates

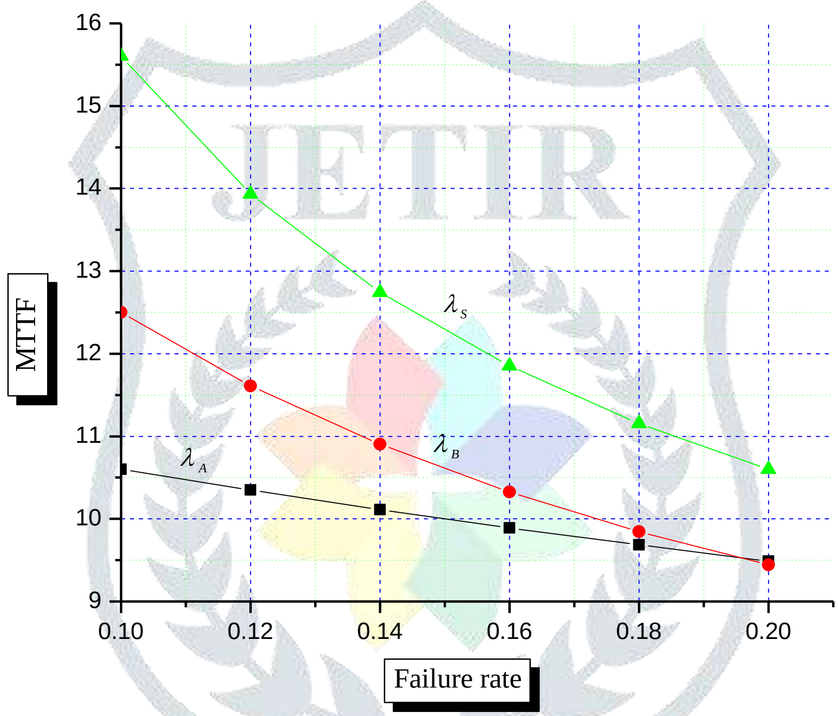

Fig. 4 System's MTTF vs. Failure rates

5.4 Sensitivity Analysis

One of the most vital requirements of system designers is to know "most/least affecting components" of the system. Sensitivity analysis does the same. Basically it is technique which identifies the components of a system which affect its performance most and least. Here in the present paper author done the sensitivity analysis for system's reliability as well as system's MTTF.

5.4.1 Sensitivity analysis for system's Reliability

It has done by using the reliability of the system which is given in equation (28). Table 6 and corresponding Fig. 5 give the insights about sensitivity of system's reliability.

Time unit "*t*"	Sensitivity of system's Reliability		
	λ_A	λ_B	λ_S
0	0	0	0
1	-0.00525750	-0.08445190	-0.06372150
2	-0.03317340	-0.28407900	-0.21581670
3	-0.08831260	-0.53552060	-0.40991200
4	-0.16513470	-0.79502540	-0.61350320
5	-0.25445450	-1.03433010	-0.80506710
6	-0.34692700	-1.23692730	-0.97149420
7	-0.43471540	-1.39488680	-1.10591040
8	-0.51209563	-1.50626537	-1.20589252
9	-0.57546914	-1.57307015	-1.27205487
10	-0.62309133	-1.59971416	-1.30695578

Table 6 Sensitivity for system's Reliability vs. Time unit "*t*"

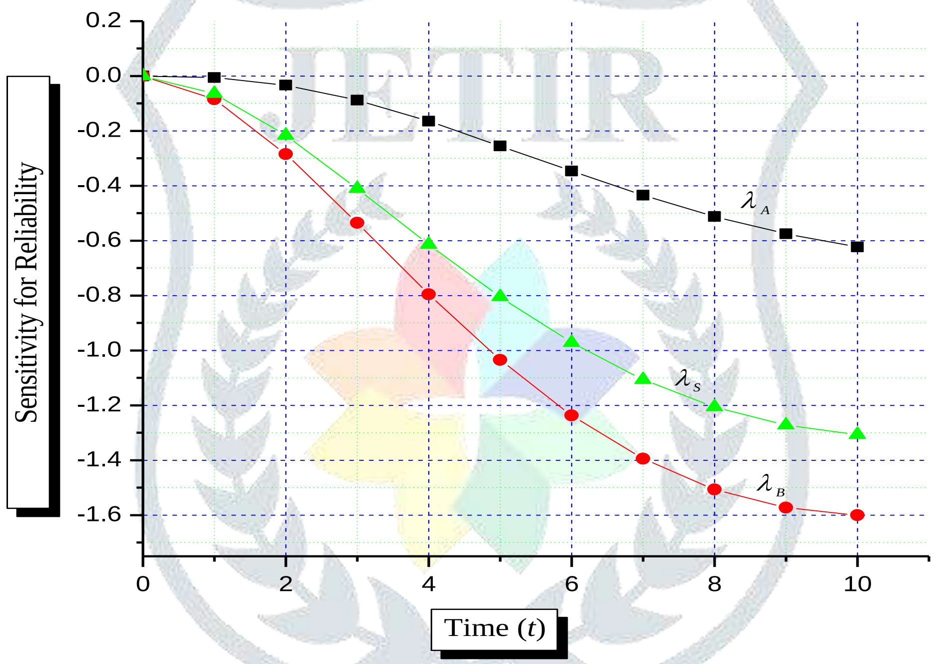

Fig. 5 Sensitivity for system's Reliability vs. Time unit "*t*"

5.4.2 Sensitivity analysis for system's MTTF

It has done by using the MTTF of the system which is given in equation (30). Table 7 and corresponding Fig. 6 give the insights about sensitivity of system's MTTF.

Failure rate	Sensitivity of system's Reliability		
	λ_A	λ_B	λ_S
0.10	-12.80000000	-50.00000000	-100.0000000
0.12	-12.19326322	-39.44402705	-69.44444445
0.14	-11.48058551	-31.82870369	-51.02040817
0.16	-10.74149911	-26.17205279	-39.06250000
0.18	-10.01753068	-21.86588922	-30.86419753
0.20	-9.329446066	-18.51851853	-25.00000000

Table 7 Sensitivity for system's MTTF vs. Failure rates

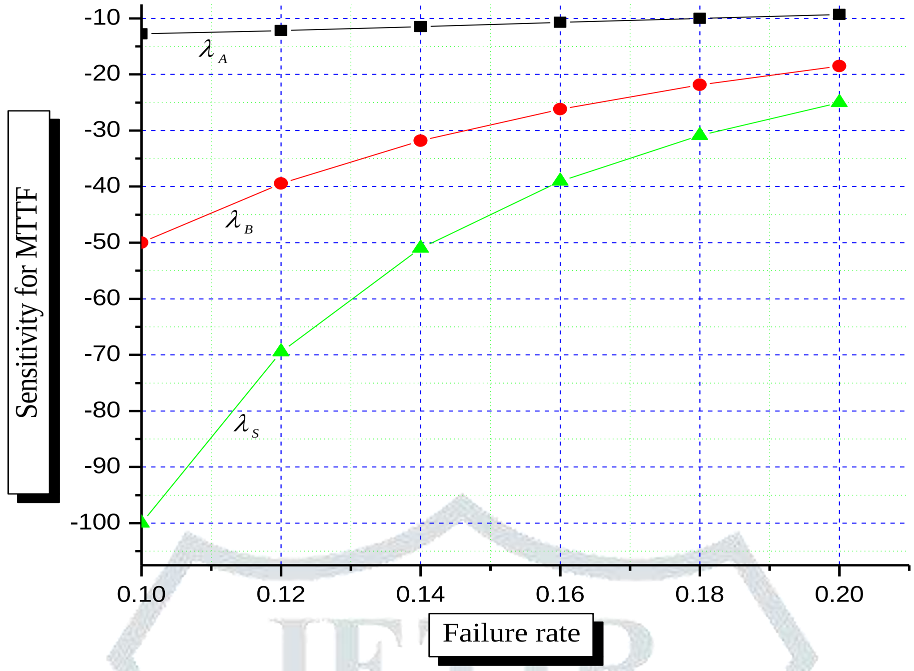

Fig. 6 Sensitivity for system's MTTF vs. Failure rates

6. Result discussion and Conclusion

On the basis of the above graphs (for different reliability measures) the author found that

The availability of the considered system is decreasing as time passes and obtained a value of 0.3884600045 (Fig. 2) at 10 units of time. Fig 3 reflects the behavior of reliability with time unit. It is observed the reliability of the system at ten units of time is 0.445998704. The considered system's MTTF is reflected in Fig. 4. It is observed that the condensate system's MTTF is highest/least with respect to failure rate of standby unit/unit A. The sensitivity for system's reliability with time is given in Fig. 5. It is observed from it the reliability of considered system is highly sensitive with respect to failure rate of unit B. Moreover, the system's MTTF is highly sensitive with respect to the failure rate of standby unit. So it can be concluded that in order to make system more reliable one have to pay more attention on the maintenance of these units. It asserts that the present research is beneficial for the management who is having the similar systems.

Reference

1. Sarhan, A. M. (2005). Reliability equivalence factors of a parallel system. *Reliability Engineering & System Safety*, *87*(3), 405-411.
2. Wang, W., & Loman, J. (2002, January). Reliability/availability of K-out-of-N system with M cold standby units. In *Annual Reliability and Maintainability Symposium. 2002 Proceedings (Cat. No. 02CH37318)* (pp. 450-455). IEEE.

3. Nezakati, E., & Razmkhah, M. (2018). On Reliability Analysis of $ k $-Out-of-$ n $: $ F $ Systems Equipped with a Single Cold Standby Component Under Degradation Performance. *IEEE Transactions on Reliability*, *67*(2), 678-687.
4. S. Eryilmaz, (2012) "On the mean residual life of a k-out-of-n:G system with a single cold standby component," Eur. J. Oper. Res., vol. 222, pp. 273–277.
5. Van Gemund, A. J., & Reijns, G. L. (2012). Reliability analysis of $ k $-out-of-$ n $ systems with single cold standby using Pearson distributions. IEEE Transactions on Reliability, 61(2), 526-532.
6. Coit, D. W. (2001). Cold-standby redundancy optimization for nonrepairable systems. Iie Transactions, 33(6), 471-478.
7. Tuteja RK, Arora RT, Taneja G (1991) Analysis of a two-unit system with partial failures and three types of repairs. Reliab Eng Syst Saf 33:199–214
8. Alidrisi MM (1992) The reliability of a dynamic warm standby redundant system of n components with imperfect switching. Microelectron Reliab 32(6):851–859
9. Chandrasekhar P, Natarajan R, Yadavalli VSS (2004) A study on a two-unit standby system with Erlangian repair time. Asia- Pac J Oper Res 21(3):271–277
10. Xu H, Guo W, Yu J, Zhu G (2005) Asymptotic stability of a repairable system with imperfect switching mechanism. Int J Math Math Sci 4:631–643