JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

Fair Support Equitable Domination in Graphs: A Novel Approach and Its Applications

Krupali Bhatt[1], Dr. Tushharkumar Bhatt[2]
1. Research Scholar, Department of Mathematics, Atmiya University, Rajkot.
krupalibhatt1612@gmail.com

2. Assistant Professor, Department of Mathematics, Atmiya University, Rajkot.
tushar.bhatt@atmiyauni.ac.in

March 30, 2025

Abstract : In this research, we introduce a novel form of domination, namely fair support equitable domination, which extends traditional graph domination concepts. These new domination parameters are designed to capture specific types of domination in graphs, providing a more nuanced understanding of how vertices influence the overall structure and connectivity of a graph. We investigated these parameters within the framework of common graph types, including paths, cycles, and complete graphs, with the aim of identifying distinctive features and characteristics of each form of domination.

Our study employs a detailed analysis of various graph classes, examining how fair support equitable domination behaves across different configurations. Through this examination, we uncover several important relationships between the structure of these graphs and the newly introduced domination parameters. Additionally, we present theoretical results that highlight key patterns and structural insights, enhancing the comprehension of the impact of these parameters on graph theory. The findings offer a fresh perspective on domination in graphs and provide a foundation for further exploration into their applications in optimization problems, network design, and other related fields.

Ultimately, this research contributes to the broader understanding of domination theory by introducing new parameters that open up new avenues for exploration and application in graph theory.

Keywords: Support equitable dominating set, Fair dominating set, Fair support equitable dominating set.

I.Introduction

Throughout the paper, we consider the graph G as finite undirected simple graphs. We use the standard graph theory notation, as in (R. Balakrishnan and K. Ranganathan , 1999). For an introduction to the theory of domination in graphs we refer to (T. W. Haynes, S. T. Hedetniemi and P. J. Slater, 1998). There are different types of domination such as connected domination, independent domination, total domination, paired domination, secure domination, etc. (R. Guruviswanathan M. A., 2017) investigated support equitable domination in graphs, while (S. Sangeetha, 2022) studied fair domination of special graphs.

II.Background

Let G be a simple finite graph with a vertex set $V(G)$ and an edge set $E(G)$.

Definition 2.1: The **open neighborhood** $N(v)$ of the vertex v is consists of the set of vertices adjacent to v that is $N(v) = \{w \in V : vw \in E\}$ and **closed neighborhood** of v is $N[v] = N(v) \cup \{v\}$.

Definition 2.2: The **degree of a vertex** v in a graph G is the number of edges incident with v and is denoted by $deg(v)$.

Definition 2.3: A set $S \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. The **domination number** $\gamma(G)$ of G equals the minimum cardinality of a dominating set S in G.

Definition 2.4: A subset S of V is **independent** if no two vertices in S are adjacent.

Definition 2.5: If every vertex of the graph has the same support, then the graph is called **support regular graph**.

Definition 2.6: If every vertex of the graph has support either k or $k+1$ then the graph is called as **bi-support regular graph**.

Definition 2.7: A subset S of V is called an equitable dominating set if for every $v \in V - S$ there exists a vertex $u \in S$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$, where $deg(u)$ and $deg(v)$ denotes the degree of a vertex u and v respectively. The minimum cardinality of such a dominating set is denoted by $\gamma^e(G)$ and is called the **equitable domination number** of G.

Definition 2.8: The **support of a vertex** in G is defined as the sum of the degree of its neighbours.

Definition 2.9: A subset S of a vertex set V of a graph G is called support equitable dominating set of G if for any $v \in V - S$, there exists a $u \in S$, such that $uv \in E(G)$ and $|supp(u) - supp(v)| \leq 1$. The minimum cardinality of a support equitable dominating set of G is called **support equitable domination number** and denoted by $\gamma_{se}(G)$.

Definition 2.10: A dominating set $S \subseteq V(G)$ is a fair dominating set in G if for every two distinct vertices u and v from $V(G) - S$, $|N(u) \cap S| = |N(v) \cap S|$, that is every two distinct vertices not in S have the same number of neighbors from S. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$ is called the **fair domination number**.

In this paper, we use this idea to develop the concept of fair support equitable dominating set and fair support equitable domination number.

III.Main Thrust of the Paper

- **Fair Support Equitable Domination**

Definition 3.1: Let $G = (V, E)$ be a simple finite graph. $D \subseteq V$ is a fair support equitable dominating set of G if for every two distinct vertices u and v from $V - D$,

i.for every $n > 1$, $|SUP(u) - SUP(v)| \leq \lceil \frac{m}{n\sqrt{n-1}} \rceil$

ii.for $n = 1$, $|SUP(u) - SUP(v)| \leq \lceil \frac{n}{m\sqrt{m-1}} \rceil$

where, $SUP(u) = \sqrt{\sum d(u_i)^2}$; $u_i \in N(u)$, $m =$ order of G (or no. of vertices), $n = |D|$ (cardinality of a fair dominating set).

The minimum cardinality of a fair support equitable dominating set of G is called **fair support equitable domination number** of G and is denoted by $\gamma_{fse}(G)$.

- **Fair Support Equitable Domination Number for Some Well-Known Graphs:**

i.$\gamma_{fse} G = 1$; where G has unique fair dominating set like $(K_n), (K_{1,n}), (W_n), (F_n)$.

ii.$\gamma_{fse}(\overline{K}_n) = n$

iii.$\gamma_{fse}(K_{m,n}) = 2$

iv.$\gamma_{fse}(P_n) = \lceil \frac{n}{3} \rceil; n \geq 2$

v.$\gamma_{fse}(C_n) = \{\lceil \frac{n}{3} \rceil ; n = 3k \& n = 3k+1 \; \lceil \frac{n}{3} \rceil + 1 ; n = 3k-1$

Below are visual representations of the aforementioned results.

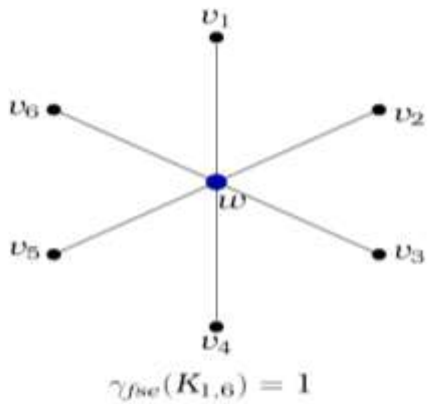

Figure 1 - Star graph

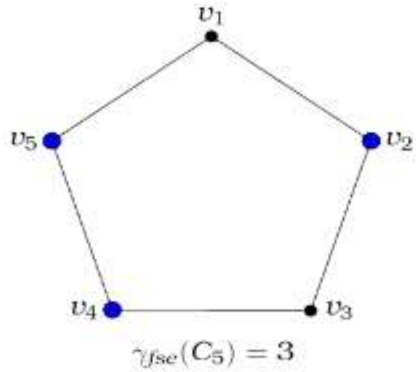

Figure 2 - Cycle graph

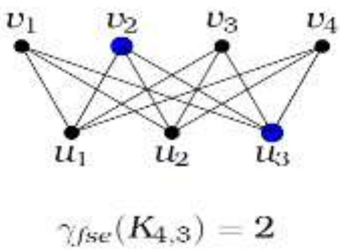

Figure 3 - Complete bipartite graph

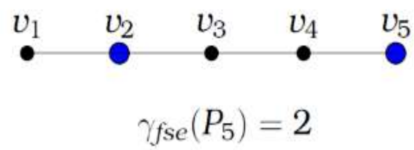

Figure 4 - Path graph

- **Fair Support Equitable Domination Number for Some Special Graphs:**

I. The n **- sunlet graph** is a graph on $2n$ vertices are obtained by attaching $n -$ pendant edges to the cycle C_n and it is denoted by S_n.

- $\gamma_{fse}(S_n) = n$ for $n \geq 3$.

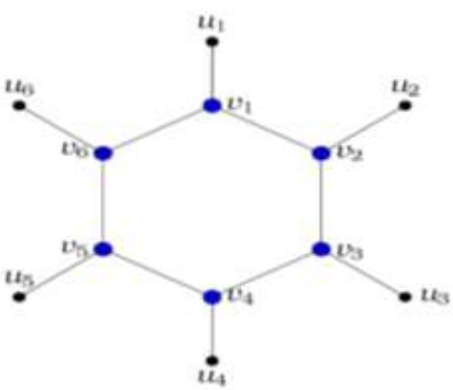

Figure 5 - 6-sunlet graph

II. The **Helm graph** H_n is the graph with $2n + 1$ vertices obtained from an $n -$wheel graph by adjoining a pendant edge at each vertex of the cycle.

- $\gamma_{fse}(H_n) = n + 1$ for $n \geq 3$.

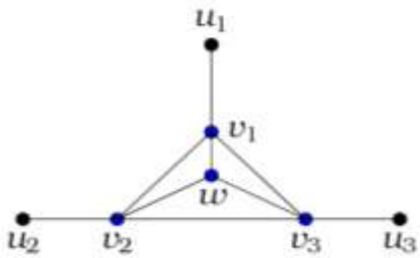

Figure 6- Helm graph H_3

III. A **crown graph** $H_{n,n}$ with $2n$ vertices and V_1, V_2 be the two partitions of a vertex set V, that is $(V = V_1 \cup V_2)$ where $|V_1| = |V_2|$ and $V_1 \cap V_2 = \phi$ are connected in such a way that each vertex $u_i \in V_1$ is connected with $v_j \in V_2$ if $i \neq j$.

- $\gamma_{fse}(H_{n,n}) = 2$ for $n \geq 3$.

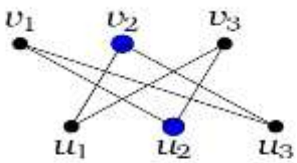

Figure 7 – Crown graph $H_{3,3}$

IV. The n **- web graph** with $3n$ vertices and V_1, V_2, V_3 be the three partition of a vertex set V i.e. $(V = V_1 \cup V_2 \cup V_3)$ such that the cycle with n vertices $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_n\}$ are connected in such a way that each vertex u_i and v_i are connected and v_i is connected with pendent vertex $w_i, i = 1,2, \ldots, n$ respectively.

- $\gamma_{fse}(n - web) = n$ for $n \geq 3$.

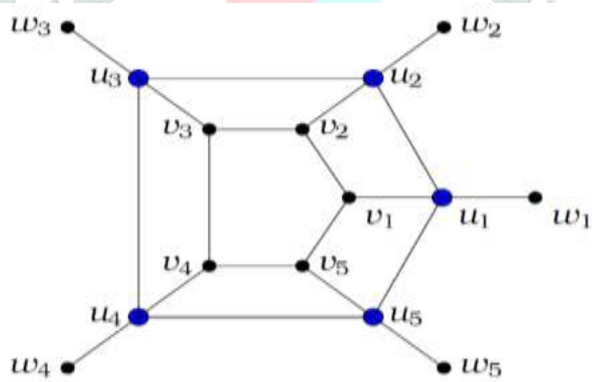

Figure 8 – (5 - web graph)

V. The **wind mill graph** $Wd(m,n)$ is the graph obtained by taking n copies of the complete graph K_m with a vertex in common where $n \geq 3$.

- $\gamma_{fse}(Wd(m,n)) = 1$ for $m \geq 2$ & $n \geq 3$.

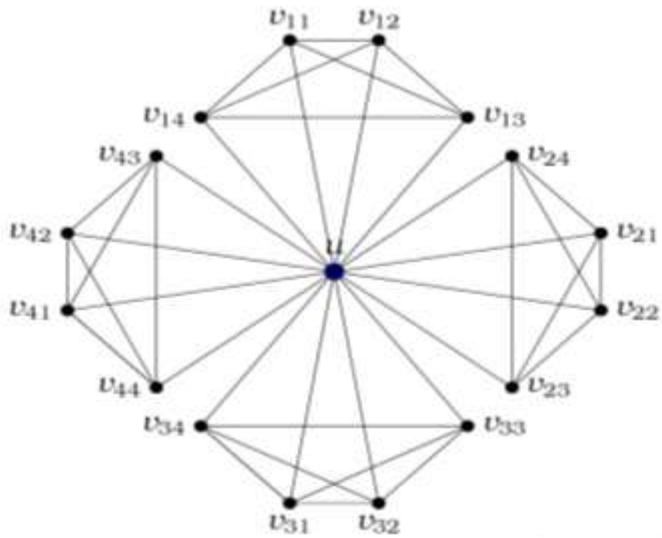

Figure 9 – Wind mill graph Wd (4,5)

VI.The **ladder graph** L_n is defined as $P_n \square P_2$ where P_n is a path with n vertices cartesian product with P_2, a path with two-vertices.

- $\gamma_{fse}(L_n) = \{\lceil\frac{2n}{3}\rceil + 1 \,; n \equiv 1(mod3)\ \lceil\frac{2n}{3}\rceil \,;\ otherwise\quad for\ n \geq 2$

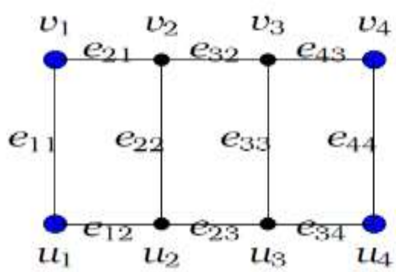

Figure 10 – Ladder graph L4

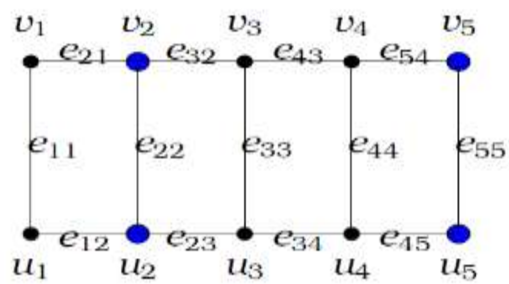

Figure 11 – Ladder graph L5

- **Some General Results on Fair Support Equitable Domination Number:**

Theorem 3.1: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two isomorphic graphs, then $\gamma_{fse}(G_1) = \gamma_{fse}(G_2)$.
Proof: Let G_1 and G_2 be two isomorphic graphs of order m_1 and m_2 i.e. $m_1 = m_2$. Suppose, if possible $\gamma_{fse}(G_1) \neq \gamma_{fse}(G_2)$. Let D_1 and D_2 be a minimum fair support equitable dominating sets of G_1 and G_2 respectively i.e. $|D_1| \leq m_1$ and $|D_2| \leq m_2$ by supposition, $|D_1| \neq |D_2| \Rightarrow m_1 \neq m_2$ Therefore, the fair support equitable domination number of isomorphic graph is equal.

Remark 3.2: Converse of the above theorem is not true. For example, $\gamma_{fse}(C_5) = \gamma_{fse}(P_9) = 3$ but C_5 & P_9 are non-isomorphic graphs.

Theorem 3.3: Let $G = (V_1, E_1)$ be the graph and $H = (V_2, E_2)$ be the connected subgraph of G, then $\gamma_{fse}(H) \leq \gamma_{fse}(G)$.
Proof: Let $D \subseteq V_1$ and $S \subseteq V_2$ be the fair support equitable dominating sets of G and H respectively. As H is a subgraph of $G, V_2 < V_1 \Rightarrow |D| < |S|$.

Remark 3.4: If H be the spanning subgraph of G, then $\gamma_{fse}(H)$ may or may not be $\leq \gamma_{fse}(G)$.
Remark 3.5: For any graph G, the fair support equitable dominating set is also a dominating set. $\therefore \gamma(G) \leq \gamma_{fse}(G)$.

Theorem 3.6: If G is support regular or bi-support regular $(k, k+1)$, for some k then $\gamma_{fse}(G) = \gamma(G)$.
Proof: Let G be a graph of order $n \geq 2. D$ be the minimum dominating set of G. $\therefore |D| = \gamma(G) \leq n$ (if G is disconnected, $\gamma(G) = n$)

Case(1): Suppose G is a support regular graph, $\forall v_i \in V(G) \ni SUP(v_i) = k$
Let $u, v \in V - D$ also $SUP(u) = SUP(v) = k \Rightarrow |SUP(u) - SUP(v)| = 0 < [\frac{n}{m\sqrt{m-1}}]$.
D is a fair support equitable dominating set of G and $|D| = \gamma(G) \geq \gamma_{fse}(G)$ but by remark 3.5 $\gamma(G) \leq \gamma_{fse}(G)$.
$\therefore \gamma(G) = \gamma_{fse}(G)$

Case(2): Suppose G is a bi-support regular graph, $\forall v_i \in V(G) \ni SUP(v_i) = k$ or $SUP(v_i) = k + 1$
Let $u, v \in V - D$ also $SUP(u) = k$ or $k + 1$ and $SUP(v) = k$ or $k + 1 \Rightarrow |SUP(u) - SUP(v)| = 0$ or $1 < [\frac{n}{m\sqrt{m-1}}]$($\because |\gamma(G)| \leq$ no. of vertices).
D is a fair support equitable dominating set of G and $|D| = \gamma(G) \geq \gamma_{fse}(G)$ but by remark 3.5$\gamma(G) \leq \gamma_{fse}(G)$.
$\therefore \gamma(G) = \gamma_{fse}(G)$

IV. Results Related to γ_{fse} of Graph Operations:

Let G and H be two non-empty graphs.

i.Union: $\gamma_{fse}(G \cup H) \geq max\{\gamma_{fse}(G), \gamma_{fse}(H)\}$

Proof. Let G be a graph with n vertices and let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of graph G. Let H be the subgraph of G and $V(H) = \{u_1, u_2, \dots, u_m\} = |V(H)| = m < n$. So, $G \cup H$ is either the graph G or the graph with more vertices and edges than G. Let $D = \{u_1, v_1, \dots v_n\}$ be the fair support equitable dominating set of $G \cup H$. Also $|D| = n$ which is maximum out of m and n.

ii.Intersection:

- If H is a subgraph of G or vice-versa, $\gamma_{fse}(G \cap H) = min\{\gamma_{fse}(G), \gamma_{fse}(H)\}$
- If G and H are two isomorphic graphs, $\gamma_{fse}(G \cap H) < min\{\gamma_{fse}(G), \gamma_{fse}(H)\}$

Proof. Let G be a graph with n vertices and let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of graph G. Let H be the subgraph of G and $V(H) = \{u_1, u_2, \dots, u_m\} = |V(H)| = m < n$. So, $G \cap H$ is the graph H. Let $D = \{v_1, u_1, \dots u_m\}$ be the fair support equitable dominating set of $G \cap H$. Also $|D| = m$ which is minimum out of m and n. If H and G are isomorphic then $G \cap H$ is the graph with less vertices and edges than H. Let $D = \{v_1, u_2, \dots u_{m-2}\}$ be the fair support equitable dominating set of $G \cap H$. Also $|D| < m$ which is less than the minimum out of m and n.

iii.Join: $\gamma_{fse}(G \vee H) \leq max\{\gamma_{fse}(G), \gamma_{fse}(H)\}$

Proof. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. and H be the subgraph of G with vertex set $V(H) = \{u_1, u_2, \dots, u_m\} = |V(H)| = m < n$. Let $D = \{w_1, v_1, \dots u_m\}$ be the fair support equitable dominating set of $G \cap H$. In the resultant graph of $G \vee H, deg(w_i) \geq deg(v_i)$ where $w_i \in V = V(G) \cup V(H)$ and $v_i \in V(G)$. $|D| \leq n$.

iv. Corona product: $\gamma_{fse}(G \odot H) \geq \gamma_{fse}(G) + \gamma_{fse}(H)$

Proof. Let G and H be two graphs with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H) = \{u_1, u_2, \dots, u_m\}$ respectively. $G \odot H$ contains n copies of H and one copy of G where i^{th} vertex of G is joined with an edge to every vertex in i^{th} copy of H. Let $D = \{v_1, v_2, \dots v_i\}$ be the fair support equitable dominating set of $G \odot H$.

v.Complement:

- If G is a disconnected graph, $\gamma_{fse}(G^{-}) < \gamma_{fse}(G)$.
- If G is a connected graph, $\gamma_{fse}(G^{-}) \geq \gamma_{fse}(G)$.

Proof: Let G be a disconnected graph with n vertices, $\gamma_{fse}(G) = n$ then G^{-} be a complete graph with n vertices are $\gamma_{fse}(G^{-}) = 1$ and vice-versa.

- **Comparison of Various Dominations:**

Types of Sets	Definition	Notation	Example and Specific Characteristics
Dominating Set (DS)	A subset $S \subseteq V(G)$ such that every vertex not in S is adjacent to at least one vertex in S.	$\gamma(G)$	Ex – 1: In a cycle graph C5, selecting two vertices such that each remaining vertex is adjacent to at least one of them. $\gamma(C_5) = 2$ Property – 1: Ensures every vertex in the graph is covered by at least one vertex in S, but does not consider balance in coverage.
Equitable Dominating Set (EDS)	A dominating set where the degree difference between any two vertices in S is at most one.	$\gamma^e(G)$	Ex – 2: In a graph where degrees are balanced, selecting nodes so that each has a similar influence. $\gamma^e(C_5) = 2$ Property – 2: Ensures degree balance within the dominating set, making the selection more uniform.
Support Equitable Dominating Set (SEDS)	A dominating set where the support difference between any two vertices in S is at most one.	$\gamma_{se}(G)$	Ex – 3: Selecting vertices in a graph where each dominating vertex supports almost the same number of non-dominating vertices. $\gamma_{se}(C_5) = 2$ Property – 3: Balances influence based on support (number of dominated vertices per vertex in S).
Fair Dominating Set (FDS)	A dominating set where all non- dominating vertices are dominated by the same number of vertices from S.	$\gamma_{fd}(G)$	Ex – 4: Choosing S such that each non-dominating vertex is connected to the same number of nodes in S. $\gamma_{fd}(C_5) = 3$ Property – 4: Focuses on fairness in coverage rather than just the size of the dominating set.
Fair Support Equitable Dominating Set (FSED)	A dominating set where the support difference between any two vertices in S is at most the relation between no of vertices of G and cardinality of S.	$\gamma_{fse}(G)$	Ex – 5: Extends FDS by ensuring that support is distributed proportionally to the overall graph structure. $\gamma_{fse}(C_5) = 3$ Property – 5: Advanced version incorporating fairness, equitable support, and balanced influence distribution, ensuring proportional coverage across the graph.

Table 1 - Comparison of Various Dominations

- **Expanded Applications and Challenges**

Dominating sets, including their variations such as equitable dominating sets, support equitable dominating sets, fair dominating sets, and support fair equitable dominating sets, have numerous real-world applications in various fields. Below are specific applications for each type of dominating set along with the problems they face and solutions that require more advanced variations.

a) Dominating Set (DS)

Applications:

- Wireless Sensor Networks (WSN): Selecting key sensor nodes to ensure full network coverage while minimizing energy consumption.
- Surveillance and Security: Optimal placement of security cameras to monitor all key locations with minimal resources.
- Urban Planning: Placing emergency service centers (fire stations, hospitals) to provide the best coverage for a city.

Problems:

- Uneven workload distribution among selected nodes.

- Some nodes may be overloaded while others remain underutilized.

Solution (Using EDS): Ensuring that the chosen dominating nodes have similar degrees to balance the workload.

b) Equitable Dominating Set (EDS)

Applications:

- Social Network Influence: Ensuring that influence in a social network is distributed fairly across different user groups.
- Load Balancing in Computing: Distributing tasks among servers in a cloud network to maintain equitable processing loads.
- Traffic Control Systems: Balancing traffic signal placement to ensure fair distribution of vehicle flow and minimize congestion.

Problems:

- Does not consider the number of connections a node supports.
- Some dominant nodes might still be overloaded compared to others.

Solution (Using SEDS): Taking into account the support difference to ensure nodes have balanced influence over others.

c) Support Equitable Dominating Set (SEDS)

Applications:

- Warehouse Distribution Networks: Ensuring fair distribution of goods among warehouses based on their service demand.
- Telecommunications Infrastructure: Fairly allocating bandwidth and network resources across different regions.
- Healthcare Resource Allocation: Ensuring fair distribution of hospital resources like ICU beds and ventilators.

Problems:

- Equitable domination does not always ensure fairness in support distribution.
- Some dominating nodes might still control significantly more non-dominating nodes than others.

Solution (Using FDS): Ensuring that every dominated node is influenced by the same number of dominating nodes to maintain fairness.

d) Fair Dominating Set (FDS)

Applications:

- Public Transportation Optimization: Ensuring equal access to public transport stops for all urban regions.
- Disaster Relief Planning: Equitably distributing aid centers to ensure fair access during emergencies.
- AI Task Allocation: Assigning tasks among autonomous agents while maintaining fairness in workload distribution.

Problems:

- Does not consider support-based fairness and workload balancing simultaneously.
- Some dominant nodes may still dominate significantly more nodes than others.

Solution (Using SFEDS): Ensuring a fair distribution of domination influence while also considering support balance.

e) Support Fair Equitable Dominating Set (SFEDS)

Applications:

- Smart Cities: Ensuring fair placement of charging stations for electric vehicles to optimize accessibility.
- Multi-Robot Systems: Balancing tasks in robotic swarms for efficient exploration and resource collection.
- Environmental Monitoring: Strategic sensor deployment for pollution and climate monitoring, ensuring fairness in geographic coverage.

Problems:

- This approach solves issues that could not be handled by previous dominating sets.
- However, the complexity of computing SFEDS is higher, requiring advanced algorithms and computational resources.

Solution: Using machine learning and optimization techniques to compute SFEDS efficiently and employing heuristic and approximation algorithms to reduce computational overhead while ensuring fairness and support balance.

1. Future Trends

Future research in fair support equitable domination can explore fairness-oriented models in dynamic and multilayer graphs, reflecting real-world complexity. This opens a rich avenue for researchers to develop algorithms that ensure balanced influence and representation across diverse networks.

2. Conclusion

In this research paper, we introduce a new variant of domination and establish key results for standard and special graphs. We analyzed its properties in isomorphic graphs, subgraphs and support regular graphs. Some applications of domination are also identified. These findings contribute to domination theory and open avenues for further exploration in graph applications.

V. References

Dr. G. Ramesh, A. S. (2018). Some new results on helm graphs,. *JETIR*, 5(11), 116-125.

R. Balakrishnan and K. Ranganathan . (1999). *A text book of graph theory.* Springer.

R. Guruviswanathan, M. A. (2017). Support equitable domination in graphs. *International journal of emerging technology and advanced engineering.*, *9*(7), 190-192.

R. Guruviswanathan, V. S. (2016). Bounds for support equitable domination number. *IJCTER*, 2(6), 11-15.

S. Sangeetha, J. U. (2022). On fair domination of special graphs. *2nd International conference on mathematical techniques and applications.* AIP Conference Proceedings 2516, 210026.

Shalini Rajendra Babu, N. R. (2019). On prime cordial labeling of crown, armed crown, H-graph and butterfly graph. *IJITEE*, *2*(9), 15-26.

T. W. Haynes, S. T. Hedetniemi and P. J. Slater. (1998). *Fundamentals of domination in graphs.* Marcel Dekker, Inc.

V. Swaminathan, D. a. (2011). Degree equitable domination in graphs. *Kragujevac journal of mathematics.*, 35(1), 191-197.

V. Swaminathan, R. S. (2021). Equitable fair domination in graphs. *Discrete Mathematics, Algorithms and Applications*, 13(6), 176-181.

Yair Caro, A. H. (2012). Fair Domination in Graphs. *Discrete Mathematics*, 312(19), 2905-2914.

Biodata

I am a Research Scholar in the Department of Mathematics at Atmiya University. I hold a Master's degree in Mathematics from Saurashtra University. My research focuses on Graph Theory, with a particular interest in the study of **Domination in graphs and its related concepts**. I have published several research papers in reputed journals and have presented my work at both national and international conferences. With a strong commitment to academic excellence, I actively engage in research collaborations and explore new developments in my field.

Research Guide:
Dr. Tushharkumar Bhatt is a distinguished professor in the Department of Mathematics at Atmiya University. With extensive expertise in Domination, he has mentored numerous research scholars and contributed significantly to the field of **Graph Theory**. His guidance and scholarly insights have played a crucial role in shaping my research.