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Paired and Inverse Paired Domination Number of Wheel Related Graphs

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Abstract: The set of edges with no common end vertex is called a matching. A matching is perfect if every vertex of the graph is incident to an edge of the matching. The subset of vertex set of a graph is a paired dominating set if it is dominating and subgraph induced by the set contains a perfect matching. The paired domination number γ_{pr} is the cardinality of a minimum paired dominating set for a graph. And the set with minimum cardinality is known as minimum paired dominating set. Let a subset D of the vertex set $V(G)$ be the minimum paired dominating set for the graph G. If the set $V(G) - D$ contains a paired dominating set D' of the graph G, then D' is an inverse paired dominating set with respect to D. The inverse paired domination number $\gamma_{pr}^{-1}(G)$ is the minimum cardinality of an inverse paired dominating set. The present paper is aimed to report some results and investigations in the context of inverse paired dominating sets in wheel related graphs.

Keywords: Paired domination, Inverse paired domination, Wheel related graphs

I. Introduction

In the paper, we begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The concept of dominating set is well studied by Berge [1] in 1958 and Ore [10] in 1962. A systematic survey on dominating sets and related concepts can be found in Haynes *et al.* [3]. The discussion on some advanced topics on the theory of domination is carried out by the same authors in [4].

The concept of inverse domination was introduced by Kulli and Sigarkanti [8] while in [9] Kulli and Nandargi have explored the concept of inverse domination and introduced some different parameters of inverse dominating sets such as inverse independent dominating set, inverse paired dominating set and inverse induced paired dominating set. For any undefined term in graph theory we refer to Clark and Holtan [2].

Paired domination number for k^{th} power of path and cycle have been investigated by Isaac and Pandya in [5] and also studied paired domination of degree splitting graph of path and cycle. Further paired domination number of some path and cycle related graphs are investigated in [6] by the same authors.

For the definition of wheel and related graphs, we have referred a paper by C. Kaithavalappil et al. [7] and a paper by Vaidya and Pandit [12].

Background

Definition 1.1. *The set of edges with no common end vertex is called a matching. A matching is perfect if every vertex of the graph is incident to some edge of the matching.*

Definition 1.2. *The subset D of a vertex set $V(G)$ of a graph G is a paired dominating set if it is a dominating set and subgraph induced by it contains a perfect matching. The paired domination number γ_{pr} is the minimum cardinality of a paired dominating set D for a graph G and D is known as minimum paired dominating set.*

Definition 1.3. *Let D be the minimum paired dominating set for a graph G. If $V(G) - D$ contains a paired dominating set D' of G, then D' is called an inverse paired dominating set with respect to D. The inverse paired domination number, denoted by $\gamma_{pr}^{-1}(G)$ of G, is the minimum cardinality of an inverse paired dominating set of G.*

Definition 1.4. *A wheel graph, denoted by W_n, is the graph obtained by joining all the vertices of a cycle C_n to a common vertex known as the apex vertex. The vertices of cycle C_n is known as rim vertices. It is also defined as $W_n = C_n + K_1$.*

***Definition 1.5**. A double wheel graph, denoted by DW_n, is the graph obtained by joining two disjoint cycles to an external vertex. It is also defined as $DW_n = 2C_n + K_1$.*

***Definition 1.6**. A gear graph, denoted by G_n, is the graph obtained from wheel graph by adding vertex between two adjacent rim vertices.*

***Definition 1.7**. A helm graph, denoted by H_n, is the graph obtained from wheel graph by attaching pendant vertex to every rim vertex.*

***Definition 1.8**. A closed helm graph, denoted by CH_n, is the graph obtained from helm graph by joining pendant vertex u_i to u_{i+1} to form an outer cycle.*

***Definition 1.9**. A flower graph, denoted by F_n, is the graph obtained from helm graph by joining each pendant vertex to the apex vertex.*

***Definition 1.10**. A sunflower graph, denoted by S_n, is the graph obtained from wheel graph by adding the vertex u_i with respect to each rim vertex v_i and join them with v_i and v_{i+1} while joining u_n with v_n and v_1.*

***Definition 1.11**. A closed sunflower graph, denoted by CS_n, is the graph obtained from sunflower graph by joining u_i to u_{i+1} and u_n to u_1 to form an outer cycle.*

***Definition 1.12**. A blossom graph, denoted by Bl_n, is the graph obtained from closed sunflower graph by joining each u_i to the apex vertex.*

The inverse paired domination number of cycle, wheel and complete bipartite graphs have been investigated by Vaidya and Pandya in [7]. They also investigated inverse paired domination number of k^{th} power of path and cycle and total graphs of path and cycle.

We state the following existing results for our ready reference.

Observations [3], [5], [11]

1. $\gamma_{pr}(C_n) = 2\lceil \frac{n}{4} \rceil$.
2. $\gamma_{pr}(W_n) = 2$.
3. $\gamma_{pr}(T(C_n)) = 2\lceil \frac{n}{3} \rceil$.
4. $\gamma_{pr}^{-1}(C_n) = \frac{n}{2}$ $where\ n \equiv 0(mod\ 4)$.
5. $\gamma_{pr}^{-1}(W_n) = 2\lceil \frac{n}{4} \rceil$.
6. $\gamma_{pr}^{-1}(T(C_n)) = 2\lceil \frac{n}{3} \rceil$.

II. Main Results

Theorem 2.1 The paired domination number $\gamma_{pr}(DW_n) = 2$.

Proof: Consider the double wheel graph with disjoint cycles as C_n. Clearly the apex vertex is adjacent to every vertex. Therefore, it dominates whole vertex set. To dominate pairwise, consider any vertex v_i from any cycle and along with apex vertex, it forms a minimum paired dominating set. Hence, $\gamma_{pr}(DW_n) = 2$.

Illustration: Consider the double wheel graph DW_8. As shown in Figure 1, the set of solid vertices represents the paired domination set and hence $\gamma_{pr}(DW_8) = 2$.

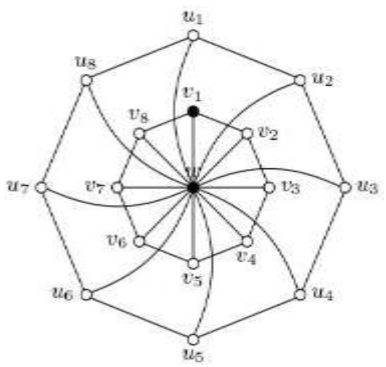

Figure 1 Paired domination set in DW_8

Theorem 2.2 For $n > 2$, the inverse paired domination number $\gamma_{pr}^{-1}(DW_n) = 4\lceil \frac{n}{4} \rceil$.

Proof: Consider the double wheel graph with disjoint cycles as C_n. Let w be the apex vertex, v_i be the vertices of first cycle and u_i be the vertices of second cycle. Consider $\{w, v_1\}$ as the minimum paired dominating set. As $n > 2$, consider the set which dominates cycle individually pairwise. Hence by known result, the inverse paired domination number $\gamma_{pr}^{-1}(DW_n) = 2\lceil\frac{n}{4}\rceil + 2\lceil\frac{n}{4}\rceil = 4\lceil\frac{n}{4}\rceil$.

Illustration: Consider the double wheel graph DW_8. As shown in Figure 2, the set of solid vertices represents the inverse paired domination set and hence $\gamma_{pr}^{-1}(DW_8) = 8$.

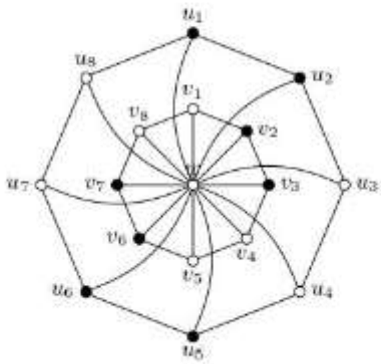

Figure 2 Inverse paired domination in DW_8

Theorem 2.3: The paired domination number $\gamma_{pr}(G_n) = 2\lceil\frac{n}{2}\rceil$.

Proof: From the definition of gear graph, it formed by adding vertex between two rim vertices of wheel graph. Therefore, the rim vertices along with newly added vertices forms a cycle with $2n$ vertices. Hence, $\gamma_{pr}(G_n) = 2\lceil\frac{2n}{4}\rceil = 2\lceil\frac{n}{2}\rceil$.

Observation: For odd n, the paired domination number for the graph G_n will be $\frac{n+1}{2}$ which is more than the number of remaining vertices and hence we cannot find an inverse paired dominating set.

Illustration: Consider the gear graph G_6. As shown in Figure 3, the set of solid vertices represents the paired domination set and hence $\gamma_{pr}(G_6) = 6$.

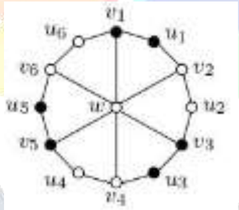

Figure 3 Paired domination set in G_6

Theorem 2.4: For even n, the inverse paired domination number $\gamma_{pr}^{-1}(G_n) = n$.

Proof: Consider even n. By the construction of gear graph, it forms a cycle of length $2n$ with alternate vertices adjacent to the apex vertex. As n is even, $2n$ is a multiple of 4. And we know $\gamma_{pr}^{-1}(C_n) = \frac{n}{2}$ where $n \equiv 0(mod\ 4)$, the inverse paired domination of gear graph, $\gamma_{pr}^{-1}(G_n) = \frac{2n}{2} = n$.

Illustration: Consider the gear graph G_6. As shown in Figure 4, the set of solid vertices represents the inverse paired domination set and hence $\gamma_{pr}^{-1}(G_6) = 6$.

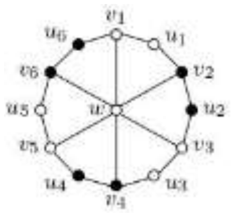

Figure 4 Inverse paired domination set in G_6

Theorem 2.5 The paired domination number $\gamma_{pr}(H_n) = \{n, n\ is\ even. n + 1, n\ is\ odd.$

Proof: From the definition of helm graph, it formed by adding pendent vertex to each rim vertices of wheel graph W_n. Consider $V(H_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are pendant vertices. If n is even, the set of vertices $\{v_1, v_2, v_3, \cdots, v_n\}$ forms a minimum pair wise dominating set while for odd n, the set of vertices $\{w, v_1, v_2, v_3, \cdots, v_n\}$ forms a minimum pair wise dominating set.

Hence, $\gamma_{pr}(H_n) = \{n, n\ is\ even. n+1, n\ is\ odd.$

Illustration: Consider the helm graph H_{10}. As shown in Figure 5, the set of solid vertices represents the paired domination set and hence $\gamma_{pr}(H_{10}) = 10$.

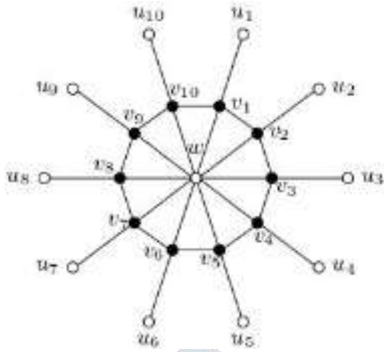

Figure 5 Paired domination set in H_{10}

Observation: The helm graph contains pendant vertices hence inverse paired domination number is not possible.

Theorem 2.6 The paired domination number $\gamma_{pr}(CH_n) = 2\left(\lceil\frac{n-1}{4}\rceil + 1\right)$.

Proof: From the definition of closed helm graph, it formed by joining all pendent vertex of helm graph to form an outer cycle. Consider $V(CH_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are vertices joined to all rim vertices. Consider the set $\{w, v_1\}$. It pairwise dominates all rim vertices and an apex vertex. To dominate the remaining vertices in outer cycle, consider it as a cycle having n vertices. Thus, we need $2\lceil\frac{n-1}{4}\rceil$ vertices. And hence $\gamma_{pr}(CH_n) = 2\left(\lceil\frac{n-1}{4}\rceil + 1\right)$.

Illustration: Consider the closed helm graph CH_{10}. As shown in Figure 6, the set of solid vertices represents the paired domination set and hence $\gamma_{pr}(CH_{10}) = 8$.

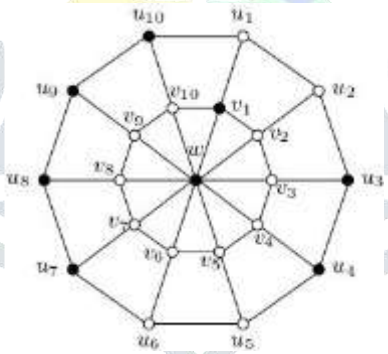

Figure 6 Paired domination in CH_{10}

Theorem 2.7 The inverse paired domination number $\gamma_{pr}^{-1}(CH_n) = \{n, n\ is\ even. n+1, n\ is\ odd.$

Proof: Consider $V(H_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are pendant vertices. Consider D be the minimum paired dominating set for closed helm graph as mentioned in previous theorem.

If n is even, consider $D' = \{u_2, v_2, v_3, v_4, \cdots, v_n\}$. Clearly, $D' \subset V - D$ and D' is a pair wise dominating set for the graph CH_n. Thus, D' is an inverse paired dominating set and it is minimum as by removal of any pair of vertices, it will not be a paired dominating set.

If n is odd, consider $D' = \{u_1, u_2, v_2, v_3, v_4, \cdots, v_n\}$. Clearly, $D' \subset V - D$ and D' is a pair wise dominating set for the graph CH_n. Thus, D' is an inverse paired dominating set and it is minimum as by removal of any pair of vertices, it will not be a paired dominating set.

Hence from both the cases, $\gamma_{pr}^{-1}(CH_n) = \{n, n\ is\ even. n+1, n\ is\ odd.$

Illustration: Consider the closed helm graph CH_{10}. As shown in Figure 7, the set of solid vertices represents the inverse paired domination set and hence $\gamma_{pr}^{-1}(CH_{10}) = 10$.

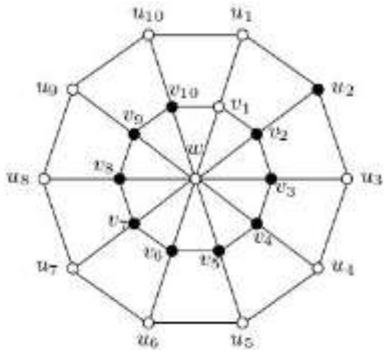

Figure 7 Inverse paired domination in CH_{10}

Theorem 2.8 The paired domination number $\gamma_{pr}(F_n) = 2$.

Proof: Consider $V(F_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are vertices joined with rim vertices and adjacent to the apex. Clearly, by the definition of flower graph, the apex vertex dominates every vertex of the graph. Therefore, to dominate it pairwise, consider any other rim vertex along with the apex and set so form dominate the graph pairwise. Hence, $\gamma_{pr}^{-1}(F_n) = 2$.

Illustration: Consider the flower graph F_{10}. As shown in Figure 8, the set of solid vertices represents the paired domination set and hence $\gamma_{pr}(F_{10}) = 2$.

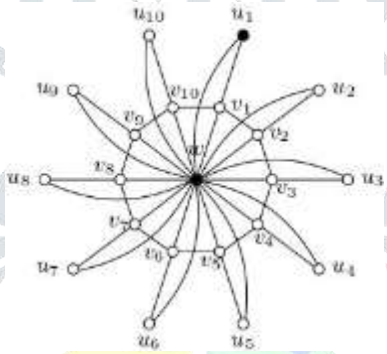

Figure 8 Paired domination set in F_{10}

Theorem 2.9 The inverse paired domination number $\gamma_{pr}^{-1}(F_n) = \{n, n \text{ is even}. n+1, n \text{ is odd}.$

Proof: Consider $V(F_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are vertices joined with rim vertices and adjacent to the apex. Consider $D = \{w, u_1\}$. It is a minimum paired dominating set for the graph F_n.

If n is even, consider $D' = \{v_1, v_2, v_3, v_4, \cdots, v_n\}$. Clearly, $D' \subset V - D$ and D' is a pair wise dominating set for the graph F_n. Thus, D' is an inverse paired dominating set and it is minimum as by removal of any pair of vertices, it will not be a paired dominating set.

If n is odd, consider $D' = \{u_1, v_1, v_2, v_3, v_4, \cdots, v_n\}$. Clearly, $D' \subset V - D$ and D' is a pair wise dominating set for the graph F_n. Thus, D' is an inverse paired dominating set and it is minimum as by removal of any pair of vertices, it will not be a paired dominating set.

Hence from both the cases, $\gamma_{pr}^{-1}(F_n) = \{n, n \text{ is even}. n+1, n \text{ is odd}.$

Illustration: Consider the flower graph F_{10}. As shown in Figure 9, the set of solid vertices represents the inverse paired domination set and hence $\gamma_{pr}^{-1}(F_{10}) = 10$.

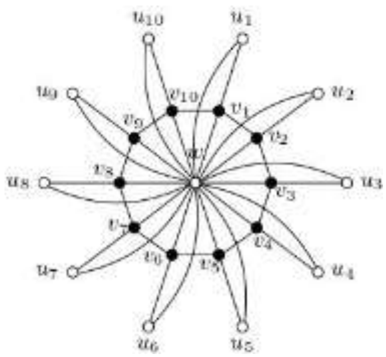

Figure 9 Inverse paired domination set in F_{10}

Theorem 2.10 The paired domination number $\gamma_{pr}(S_n) = 2\lceil\frac{n}{3}\rceil$.

Proof: Consider $V(S_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are vertices joined with rim vertices to form a sunflower graph. Consider any two adjacent vertices v_i and v_j. Clearly, it dominates two adjacent rim vertices and three $u_i's$. Therefore, to dominate all vertices, we need to consider $2\lceil\frac{n}{3}\rceil$ vertices in a paired dominating set. Hence, $\gamma_{pr}^{-1}(S_n) = 2\lceil\frac{n}{3}\rceil$.

Illustration: Consider the sunflower graph S_{12}. As shown in Figure 10, the set of solid vertices represents the paired domination set and hence $\gamma_{pr}(S_{12}) = 8$.

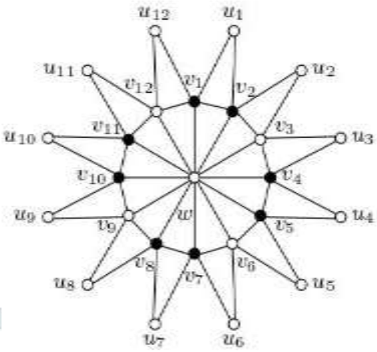

Figure 10 Paired domination set in S_{12}

Observation: For sunflower graphs, we cannot find inverse paired dominating set.

Theorem 2.11 The paired domination number $\gamma_{pr}(CS_n) = 2\left(\lceil\frac{n-2}{4}\rceil + 1\right)$.

Proof: Consider $V(CS_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are vertices adjacent to rim vertices and joined to form an outer cycle to form a closed sunflower graph. Consider the set $\{w, v_1\}$. It pairwise dominates all rim vertices, u_1, u_n and an apex vertex. To dominate the remaining vertices in outer cycle, consider it as a cycle having n vertices. Thus, we need $2\lceil\frac{n-2}{4}\rceil$ vertices. And hence $\gamma_{pr}(CS_n) = 2\left(\lceil\frac{n-2}{4}\rceil + 1\right)$.

Illustration: Consider the closed sunflower graph CS_{12}. As shown in Figure 11, the set of solid vertices represents the paired domination set and hence $\gamma_{pr}(CS_{12}) = 8$.

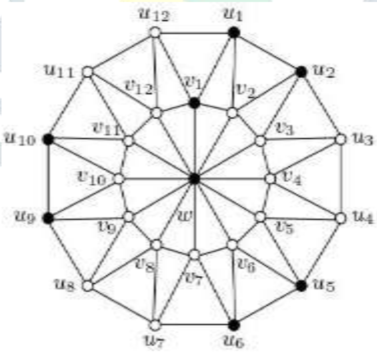

Figure 11 Paired domination set in CS_{12}

Theorem 2.12 The inverse paired domination number $\gamma_{pr}^{-1}(CS_n) = 2\lceil\frac{n}{3}\rceil$

Proof: Consider $V(CS_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are vertices adjacent to rim vertices and joined to form an outer cycle to form a closed sunflower graph. By the construction of graph, the closed sunflower graph is same as total graph of cycle $T(C_n)$ but without the apex vertex. As we consider apex vertex in a paired dominating set of the graph, we cannot consider it in inverse paired dominating set and hence inverse paired dominating set of closed sunflower graph is same as inverse paired dominating set of total graph of cycle.

Therefore, $\gamma_{pr}^{-1}(CS_n) = \gamma_{pr}(T(C_n)) = 2\lceil\frac{n}{3}\rceil$.

Illustration: Consider the closed sunflower graph CS_{12}. As shown in Figure 12, the set of solid vertices represents the inverse paired domination set and hence $\gamma_{pr}^{-1}(CS_{12}) = 8$.

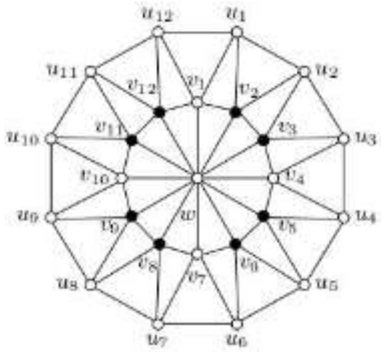

Figure 12 Inverse paired domination set in CS_{12}

Theorem 2.13 The paired domination number $\gamma_{pr}(Bl_n) = 2$.

Proof: Consider $V(Bl_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are vertices adjacent to rim vertices and joined to form an outer cycle as well as adjacent to the apex vertex to form a blossom graph. Consider the set $\{w, v_1\}$. It pairwise dominates all vertices. Therefore, it is a minimum paired dominating set and hence, $\gamma_{pr}(Bl_n) = 2$.

Illustration: Consider the blossom graph Bl_{12}. As shown in Figure 13, the set of solid vertices represents the paired domination set and hence $\gamma_{pr}(Bl_{12}) = 2$.

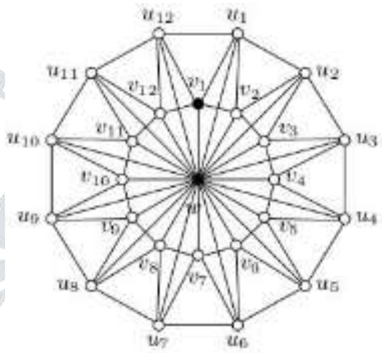

Figure 13 Paired domination set in Bl_{12}

Theorem 2.14 The inverse paired domination number $\gamma_{pr}^{-1}(Bl_n) = 2\lceil\frac{n}{3}\rceil$

Proof: Consider $V(Bl_n) = \{w, v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3, \cdots, u_n\}$ where w is the apex vertex, $v_i's$ are rim vertices and $u_i's$ are vertices adjacent to rim vertices and joined to form an outer cycle as well as adjacent to the apex vertex to form a blossom graph. Consider $\{w, v_1\}$ as minimum paired dominating set. Now to form an inverse paired dominating set, we can not consider the apex vertex. Therefore, the set will be same as the set we have considered to dominate the sunflower graph. Hence, $\gamma_{pr}^{-1}(Bl_n) = \gamma_{pr}(SF_n) = 2\lceil\frac{n}{3}\rceil$.

Illustration: Consider the blossom graph Bl_{12}. As shown in Figure 14, the set of solid vertices represents the inverse paired domination set and hence $\gamma_{pr}^{-1}(Bl_{12}) = 8$.

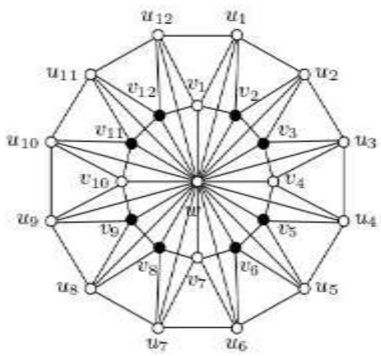

Figure 14 Inverse paired domination in Bl_{12}

Future Trends

The concept of paired domination is very much useful to restore the communication network between two vertices(nodes) while the concept of inverse paired domination helps in creation of backup plan which enhances security of a network. We have investigated the exact values of paired and inverse paired domination number of some wheel related graphs. To investigate the bounds or exact values of paired and inverse paired domination number for various graph families is an open area of research. One can investigate the family of graphs for which paired and inverse paired domination remains same.

Conclusion

In the present paper, we have investigated the exact paired as well as inverse paired domination number of some wheel related graphs. Here we have included graphs like double wheel graph (DW_n), gear graph (G_n), helm graph (H_n), closed helm graph (CH_n), flower graph (F_n), sunflower graph (S_n), closed sunflower graph (CS_n) and blossom graph (Bl_n).

References

[1] C. Berge, Theory of Graphs, Dunod, Paris (1958).

[2] J. Clark and D. A. Holtan, A First Look at Graph Theory, Allied Publishers Ltd., (1995).

[3] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New York, 1998.

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in Graphs (Advanced Topics)*, Marcel Dekker Inc., New York, 1998.

[5] R. Isaac and P. Pandya, On Paired Domination of Some Graphs, Journal of Computational Mathematica 6(2), 116-120 (2022).

[6] R. Isaac and P. Pandya, Few Results on Paired Domination Number of Some Graphs, Solovyov Studies, 71(12), 50-54 (2023).

[7] C. Kaithavalappil, S. Engoor and S. Naduvath, On Equitable Coloring Parameters of Certain Wheel Related Graphs, Contemporary Studies in Discrete Mathematics, 1(1), 3-12 (2017).

[8] V. R. Kulli and S. C. Sigarkanti, Inverse Domination in Graphs, Nat Acad. Sci. Lett., 14, $473-475$(1991).

[9] V. R. Kulli and N. R. Nandargi, Inverse Domination and Some New Parameters, Advances in Domination Theory, 1, 15-24 (2012).

[10] O. Ore, Theory of Graphs, American Mathematical Society, Colloquium Publications, (1962).

[11] S. K. Vaidya and P. B. Pandya, On Inverse Paired Domination in Some Graphs, Journal of Xidian University, 18(9), 451-460 (2024).

[12] S. K. Vaidya and R. M. Pandit, Independent Domination in Some Wheel Related Graphs, Applications and Apllied Mathematics: an International Journal, 11(1), 397-407 (2016).

Biodata

The first author, Parashree Pandya, is currently pursuing her Ph.D. since September, 2021 at the department of Mathematics, Atmiya University, Rajkot under the guidance of the second author Dr. Nirav B. Vyas. She has published three papers and one given for the publication. She has pursued her M.Sc. in Mathematics from the same university and secured gold medal in the same. She has cleared Prof. A. R. Rao examinations twice (2018 & 2020) and also secured 5th rank in west regions of India in Madhva Mathematics competition. She has also secured 2nd rank twice (2018 & 2019) in national level conference presentation held at the Christ college, Rajkot. She is an active member of GGM (Gujarat Ganit Mandal) and has delivered talk thrice (2018, 2019 & 2023) in the conferences organised by GGM. She is also an active member of ADMA (Academy of Discrete Mathematics and Applications) and presented paper twice (2023 & 2024) in the conferences organised by ADMA.